The Mean Value Theorem

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November 13, 2000

The Mean Value Theorem (MVT) is the most important theorem about derivatives. It is needed to prove some of the most commonly used facts about differentiable functions that you are familiar with from calculus. For example, it is needed to prove that if \( f : (a, b) \rightarrow \mathbb{R} \) and \( f'(x) \geq 0 \) for all \( x \in (a, b) \), then \( f \) is monotone increasing on \( (a, b) \).

Before proving the MVT (Theorem 3), we will first prove Rolle’s Theorem (which will be used in proving the MVT).

**Theorem 1 (Rolle’s Theorem)** Suppose that \( a \) and \( b \) are real numbers with \( a < b \) and suppose that the function \( f : [a, b] \rightarrow \mathbb{R} \) is differentiable at each point of \( (a, b) \) and continuous at each point of \([a, b]\). Suppose also that \( f(a) = f(b) \). Then there exists a point \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.** First, we consider the possibility that \( f \) is constant on \([a, b]\). In this case, we have \( f'(x) = 0 \) for all \( x \in (a, b) \) and, hence, the conclusion of the theorem is obviously true in this case.

Next, suppose that \( f \) is not constant on \([a, b]\). Then, either there exists a point \( d \in (a, b) \) such that \( f(d) > f(a) \) or there exists a point \( d \in (a, b) \) such that \( f(d) < f(a) \). Let’s suppose that there exists a point \( d \in (a, b) \) such that \( f(d) > f(a) \). (The other possibility is handled similarly.) In this case, since \( f \) is continuous on \([a, b]\), we know that there exists a point \( c \in (a, b) \) such that \( f(c) \geq f(x) \) for all \( x \in [a, b] \). Since \( c \in (a, b) \), we know that \( f \) is differentiable at \( c \). Also,

\[
\frac{f(x) - f(c)}{x - c} \leq 0 \text{ for all } x > c
\]

and

\[
\frac{f(x) - f(c)}{x - c} \geq 0 \text{ for all } x < c
\]
which means that

\[ f'(c) = \lim_{{x \to c^+}} \frac{f(x) - f(c)}{x - c} \leq 0 \]

and

\[ f'(c) = \lim_{{x \to c^-}} \frac{f(x) - f(c)}{x - c} \geq 0. \]

Since \(0 \leq f'(c) \leq 0\), we conclude that \(f'(c) = 0\). \(\blacksquare\)

**Exercise 2** The hypotheses of Rolle’s Theorem are that \(f : [a, b] \to R\) and

1. \(f\) is differentiable at each point of \((a, b)\).
2. \(f\) is continuous at each point of \([a, b]\).
3. \(f(a) = f(b)\)

Give examples which show that each of these hypotheses is needed in order for Rolle’s Theorem to be true. That is, give examples of functions which satisfy only two out of the three above hypotheses and for which the conclusion of Rolle’s Theorem is not true.

**Theorem 3 (Mean Value Theorem)** Suppose that \(a\) and \(b\) are real numbers with \(a < b\) and suppose that the function \(f : [a, b] \to R\) is differentiable at each point of \((a, b)\) and continuous at each point of \([a, b]\). Then there exists a point \(c \in (a, b)\) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]

**Proof.** Consider the function \(F : [a, b] \to R\) defined by

\[ F(x) = f(x) - f(a) - \left( \frac{f(b) - f(a)}{b - a} \right) (x - a). \]

Clearly, \(F\) is differentiable at each point of \((a, b)\) and continuous at each point of \([a, b]\). Also, \(F(a) = 0 = F(b)\). Thus, by Rolle’s Theorem, there exists a point \(c \in (a, b)\) such that \(F'(c) = 0\). Since

\[ F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \quad \text{for all} \; x \in (a, b), \]

\[ F'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{by Rolle’s Theorem}, \]

hence

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]
we have

\[ 0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \]

from which we conclude that

\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]


\[ \blacksquare \]

**Exercise 4**  The hypotheses of the Mean Value Theorem are that \( f : [a, b] \rightarrow R \) and

1. \( f \) is differentiable at each point of \((a, b)\).

2. \( f \) is continuous at each point of \([a, b]\).

Give examples which show that each of these hypotheses is needed in order for the Mean Value Theorem to be true. That is, give examples of functions which satisfy only one out of the two above hypotheses and for which the conclusion of the Mean Value Theorem is not true.

One of the most important corollaries of the MVT is the following:

**Corollary 5**  Suppose that \( f : (a, b) \rightarrow R \) is differentiable at each point of \((a, b)\) and suppose that \( f'(x) = 0 \) for all \( x \in (a, b) \). Then \( f \) is constant on \((a, b)\).

**Proof.** Let \( x_0 \) be a point in \((a, b)\) and let \( x \) be a point in \((a, b)\) with \( x_0 < x \). Since \( f \) is differentiable at each point of \((x_0, x)\) and continuous at each point of \([x_0, x]\), then by the MVT there exists a point \( c \in (x_0, x) \) such that

\[ f'(c) = \frac{f(x) - f(x_0)}{x - x_0}. \]

Since \( f'(c) = 0 \), then \( f(x) = f(x_0) \). Since \( x \) was chosen arbitrarily from \((x_0, b)\), this shows that \( f(x) = f(x_0) \) for all \( x \in (x_0, b) \). By similar reasoning, we can show that \( f(x) = f(x_0) \) for all \( x \in (a, x_0) \). We conclude that \( f(x) = f(x_0) \) for all \( x \in (a, b) \), i.e., \( f \) is constant on \((a, b)\). \( \blacksquare \)
Exercise 6 Suppose that \( f : (a, b) \rightarrow R \) and \( g : (a, b) \rightarrow R \) are differentiable at each point of \((a, b)\) and suppose that \( f'(x) = g'(x) \) for all \( x \in (a, b)\). Prove that there exists a constant, \( C \), such that \( f(x) = g(x) + C \) for all \( x \in (a, b)\). (In other words, prove the familiar result from calculus that if two functions have the same derivative over an interval \((a, b)\), then these two functions differ by a constant on \((a, b)\).)

Another familiar corollary of the MVT is:

Corollary 7 Suppose that \( f : (a, b) \rightarrow R \) is differentiable at each point of \((a, b)\) and suppose that \( f'(x) \geq 0 \) for all \( x \in (a, b)\). Then \( f \) is monotone increasing on \((a, b)\).

Proof. We must show that if \( x_1 \) and \( x_2 \) are any two points in \((a, b)\) with \( x_1 < x_2 \), then \( f(x_1) \leq f(x_2) \). To this end, let \( x_1 \) and \( x_2 \) be two points in \((a, b)\) with \( x_1 < x_2 \). Since \( f \) is differentiable on \((x_1, x_2)\) and continuous on \([x_1, x_2]\), then by the MVT there exists a point \( c \in (x_1, x_2) \) such that

\[
 f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\]

Since \( f'(c) \geq 0 \) and \( x_2 - x_1 > 0 \), it must also be true that \( f(x_2) - f(x_1) \geq 0 \), i.e., \( f(x_1) \leq f(x_2) \). \( \blacksquare \)

Exercise 8 Suppose that \( f : (a, b) \rightarrow R \) is differentiable at each point of \((a, b)\) and suppose that \( f'(x) \leq 0 \) for all \( x \in (a, b)\). Prove that \( f \) is monotone decreasing on \((a, b)\).

Exercise 9 Let \( I \) be an interval (of any type, possibly even \((-\infty, \infty)\)) and suppose that \( f : I \rightarrow R \) is differentiable at each point of \( I \) and that \( f' \) is bounded on \( I \) (i.e., there exists \( M > 0 \) such that \( |f'(x)| \leq M \) for all \( x \in I \)). Show that \( f \) is Lipschitzian (and hence uniformly continuous).

Exercise 10 Use the result of the previous exercise to show that the function \( f : R \rightarrow R \) given by \( f(x) = x/(1 + x^2) \) is Lipschitzian.

Exercise 11 As we have seen, there exist functions \( f : R \rightarrow R \) which are differentiable at each point of \( R \) but for which \( f' \) is not continuous at each point of \( R \). An example is the function

\[
 f(x) = \begin{cases} 
 x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
 0 & \text{if } x = 0
\end{cases}
\]

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which is differentiable at all points in \( R \) but for which \( f' \) is not continuous at 0.

Surprisingly, even though existence of \( f' \) on \( R \) does not imply continuity of \( f' \) on \( R \), existence of \( f' \) on \( R \) \textbf{does} imply that \( f' \) has the “intermediate value property” on \( R \). Specifically, if \( f : R \to R \) is differentiable at each point of \( R \), \( a \) and \( b \) are real numbers with \( a < b \) and \( f'(a) < f'(b) \), and if \( C \) is any real number with \( f'(a) < C < f'(b) \), then there exists a point \( c \in (a, b) \) such that \( f'(c) = C \). Prove this.

\textbf{Exercise 12}  Suppose that \( f : R \to R \) and suppose that there exists \( K > 0 \) such that \( \left| f(x) - f(y) \right| \leq K (x - y)^2 \) for all \( x \) and \( y \in R \). Prove that \( f \) \textbf{is constant}.

\textbf{Exercise 13}  In the textbook, Section 4.3, do problems 3, 4, and 14.