Approximation by Polynomials

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1 Second and Higher Derivatives

If \( f : (a, b) \rightarrow R \) is differentiable at all points \( x \in (a, b) \) and \( x_0 \in (a, b) \), then it makes sense to inquire about the existence of the limit

\[
\lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}
\]

If this limit exists, then \( f' \) is differentiable at \( x_0 \) and

\[
(f')'(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}.
\]

We call \((f')'(x_0)\) the second derivative of \( f \) at \( x_0 \) and denote it by \( f''(x_0) \) or by \( f^{(2)}(x_0) \).

Similarly, if \( f' \) is differentiable at all points \( x \in (a, b) \) (in which case we say that \( f \) is twice differentiable at all points \( x \in (a, b) \)), then it makes sense to inquire about the existence of the limit

\[
\lim_{x \to x_0} \frac{f''(x) - f''(x_0)}{x - x_0}.
\]

If this limit exists, then \( f'' \) is differentiable at \( x_0 \) and

\[
(f'')'(x_0) = \lim_{x \to x_0} \frac{f''(x) - f''(x_0)}{x - x_0}.
\]

We call \((f'')'(x_0)\) the third derivative of \( f \) at \( x_0 \) and denote it by \( f'''(x_0) \) or by \( f^{(3)}(x_0) \).
These definitions can be extended inductively to define higher order derivatives. In general, if \( f \) is \( n \) times differentiable at all points \( x \in (a, b) \) and if

\[
\lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}
\]

exists, then we say that \( f \) is \( n + 1 \) times differentiable at \( x_0 \) and we call

\[
f^{(n+1)}(x_0) = \lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}
\]

the \((n + 1)\)st derivative of \( f \) at \( x_0 \).

**Example 1** Consider the function \( f(x) = 8x^3 - 36x^2 + 50x - 20 \). Its first and higher derivatives at any point \( x \in R \) are

\[
\begin{align*}
    f^{(1)}(x) &= 24x^2 - 72x + 50 \\
    f^{(2)}(x) &= 48x - 72 \\
    f^{(3)}(x) &= 48 \\
    f^{(4)}(x) &= 0
\end{align*}
\]

and in fact \( f^{(m)}(x) = 0 \) for all \( n \geq 4 \).

**Example 2** For the function \( f(x) = e^x \), we have \( f^{(n)}(x) = e^x \) for all \( n \geq 1 \).

## 2 Quadratic and Higher Order Approximations

A fact that we have found to be very useful is that if \( f : (a, b) \to R \), \( x_0 \in (a, b) \), and \( f \) is differentiable at \( x_0 \), then \( f \) is approximately equal to a linear function for values of \( x \) near \( x_0 \). More specifically, there exists a function \( \alpha_1 : (a, b) \to R \) such that \( \lim_{x \to x_0} \alpha_1(x) = \alpha_1(x_0) = 0 \) and such that

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha_1(x)(x - x_0) \quad \text{for all } x \in (a, b).
\]
Solving equation (1) for \( \alpha_1 (x) \) and using the fact that \( \alpha_1 (x_0) = 0 \), we obtain
\[
\alpha_1 (x) = \begin{cases} 
\frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{x-x_0} & \text{if } x \neq x_0 \\
0 & \text{if } x = x_0
\end{cases}
\]

Now, let us suppose that \( f \) is differentiable at all points \( x \in (a, b) \) and that \( f \) is twice differentiable at \( x_0 \). We may then apply L’Hospital’s Rule to obtain
\[
\lim_{x \to x_0} \frac{\alpha_1 (x) - \alpha_1 (x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{(x-x_0)^2} = \lim_{x \to x_0} \frac{f''(x) - f''(x_0)}{2(x-x_0)} = \frac{1}{2} f'''(x_0)
\]

which shows that \( \alpha_1 \) is differentiable at \( x_0 \) and that \( \alpha_1' (x_0) = f'''(x_0) / 2 \). Hence, there exists a function \( \alpha_2 : (a, b) \to \mathbb{R} \) such that \( \lim_{x \to x_0} \alpha_2 (x) = \alpha_2 (x_0) = 0 \) and such that
\[
\alpha_1 (x) = \frac{f'''(x_0)}{2!} (x-x_0) + \alpha_2 (x) (x-x_0) \text{ for all } x \in (a, b).
\]

(We have written \( 2! \) instead of 2 in anticipation of a pattern that we will see develop.) Substitution into equation (1) gives
\[
f(x) = f(x_0) + f'(x_0) (x-x_0) + \frac{f'''(x_0)}{2!} (x-x_0)^2 + \alpha_2 (x) (x-x_0)^2 \text{ for all } x \in (a, b).
\]

This shows that \( f \) is approximately equal to a quadratic function for values of \( x \) near \( x_0 \).

Continuing with this line of reasoning, suppose that \( f \) is twice differentiable at all points \( x \in (a, b) \) and three times differentiable at \( x_0 \). Solving for \( \alpha_2 (x) \) and using the fact that \( \alpha_2 (x_0) = 0 \), we obtain
\[
\alpha_2 (x) = \begin{cases} 
\frac{f(x) - f(x_0) - f'(x_0)(x-x_0) - f''(x_0)(x-x_0)^2}{(x-x_0)^2} & \text{if } x \neq x_0 \\
0 & \text{if } x = x_0
\end{cases}
\]
Using L’Hospital’s Rule, we obtain

\[
\lim_{x \to x_0} \frac{\alpha_2 (x) - \alpha_2 (x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f (x) - f (x_0) - f' (x_0) (x - x_0) - \frac{f'' (x_0)}{2!} (x - x_0)^2}{(x - x_0)^3} \\
= \lim_{x \to x_0} \frac{f' (x) - f' (x_0) - f'' (x_0) (x - x_0)}{3 (x - x_0)^2} \\
= \lim_{x \to x_0} \frac{f'' (x) - f'' (x_0)}{3! (x - x_0)} \\
= \frac{f'' (x_0)}{3!}
\]

which shows that \( \alpha_2 \) is differentiable at \( x_0 \) and that \( \alpha'_2 (x_0) = f'' (x_0) / 3! \). Hence, there exists a function \( \alpha_3 : (a, b) \to R \) such that \( \lim_{x \to x_0} \alpha_3 (x) = \alpha_3 (x_0) = 0 \) and such that

\[
\alpha_2 (x) = \frac{f'' (x_0)}{3!} (x - x_0) + \alpha_3 (x) (x - x_0) \text{ for all } x \in (a, b).
\]

Substitution into (2) gives

\[
f (x) = f (x_0) + \sum_{j=1}^{3} \frac{f^{(j)} (x_0)}{j!} (x - x_0)^j + \alpha_3 (x) (x - x_0)^3 \text{ for all } x \in (a, b).
\]

Hence, \( f \) is approximately equal to a polynomial of degree three for values of \( x \) close to \( x_0 \).

The following theorem extends these results to the general situation of approximating a function (locally) with a polynomial of any degree.

**Theorem 3** Suppose that \( f : (a, b) \to R \) is \( n \) times differentiable at all points \( x \in (a, b) \). Also, suppose that \( x_0 \in (a, b) \) and that \( f \) is \( n + 1 \) times differentiable at \( x_0 \). Then there exists a function \( \alpha_{n+1} : (a, b) \to R \) such that \( \lim_{x \to x_0} \alpha_{n+1} (x) = \alpha_{n+1} (x_0) = 0 \) and such that for all \( x \in (a, b) \), we have

\[
f (x) = f (x_0) + \sum_{j=1}^{n+1} \frac{f^{(j)} (x_0)}{j!} (x - x_0)^j + \alpha_{n+1} (x) (x - x_0)^{n+1}.
\]
Proof. We have already proved (in the discussion preceding this theorem) that the statement of the theorem is true for \( n = 1 \). We will now show that the truth of the statement for \( n = k \) implies its truth for \( n = k + 1 \).

Assuming the statement to be true for \( n = k \), suppose that \( f \) is \( k+1 \) times differentiable at all points \( x \in (a, b) \) and that \( f \) is \( k + 2 \) times differentiable at \( x_0 \). Since the statement of the theorem is assumed to be true for \( n = k \), there exists a function \( \alpha_{k+1} : (a, b) \to R \) such that \( \lim_{x \to x_0} \alpha_{k+1} (x) = \alpha_{k+1} (x_0) = 0 \) and such that for all \( x \in (a, b) \), we have

\[
f (x) = f (x_0) + \sum_{j=1}^{k+1} \frac{f^{(j)} (x_0)}{j!} (x - x_0)^j + \alpha_{k+1} (x) (x - x_0)^{k+1} \quad \text{(3)}
\]

This gives us

\[
\alpha_{k+1} (x) = \begin{cases} \frac{f(x) - f(x_0) - \sum_{j=1}^{k+1} \frac{f^{(j)} (x_0)}{j!} (x - x_0)^j}{(x - x_0)^{k+1}} & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}
\]

By applying L’Hospital’s Rule, we obtain

\[
\begin{align*}
\lim_{x \to x_0} \frac{\alpha_{k+1} (x) - \alpha_{k+1} (x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{f (x) - f (x_0) - \sum_{j=1}^{k+1} \frac{f^{(j)} (x_0)}{j!} (x - x_0)^j}{(x - x_0)^{k+2}} \\
&= \lim_{x \to x_0} \frac{f' (x) - f' (x_0) - \sum_{j=2}^{k+1} \frac{f^{(j)} (x_0)}{(j-1)!} (x - x_0)^{j-1}}{(k + 2) (x - x_0)^{k+1}} \\
&= \lim_{x \to x_0} \frac{f'' (x) - f'' (x_0) - \sum_{j=3}^{k+1} \frac{f^{(j)} (x_0)}{(j-2)!} (x - x_0)^{j-2}}{(k + 2) (k + 1) (x - x_0)^k} \\
&\vdots \\
&= \lim_{x \to x_0} \frac{f^{(k)} (x) - f^{(k)} (x_0) - \sum_{j=k}^{k+1} \frac{f^{(j)} (x_0)}{(j-k)!} (x - x_0)^{j-k}}{(k + 2) (k + 1) \cdots (3) (x - x_0)^2} \\
&= \lim_{x \to x_0} \frac{f^{(k+1)} (x) - f^{(k+1)} (x_0)}{(k + 2)! (x - x_0)} \\
&= \frac{f^{(k+2)} (x_0)}{(k + 2)!}
\end{align*}
\]

which shows that \( \alpha_{k+1} \) is differentiable at \( x_0 \) and that \( \alpha'_{k+1} (x_0) = f^{(k+2)} (x_0) / (k + 2)! \).

Hence, there exists a function \( \alpha_{k+2} : (a, b) \to R \) such that \( \lim_{x \to x_0} \alpha_{k+2} (x) = \)
\( \alpha_{k+2} (x_0) = 0 \) and such that

\[
\alpha_{k+1} (x) = \frac{f^{(k+2)} (x_0)}{(k+2)!} (x - x_0) + \alpha_{k+2} (x) (x - x_0) \quad \text{for all } x \in (a, b).
\]

Substitution into equation (3) gives the desired result. ■

Example 4 The function \( f(x) = 8x^3 - 36x^2 + 50x - 20 \) has derivatives of all orders at all points in \((\infty, \infty)\). Suppose we want to approximate \( f \) with a third degree polynomial at the point \( x_0 = 1 \). By Theorem 3, there exists a function \( \alpha_3 \) with \( \lim_{x \to 1} \alpha_3 (x) = \alpha_3 (1) = 0 \) such that for all \( x \in (\infty, \infty) \),

\[
f(x) = f(1) + f'(1) (x - 1) + \frac{f''(1)}{2!} (x - 1)^2 + \frac{f'''(1)}{3!} (x - 1)^3 + \alpha_3 (x) (x - 1)^3.
\]

Since \( f(1) = 2, f'(1) = 2, f''(1) = -24, \) and \( f'''(1) = 48 \), we obtain

\[
f(x) = 2 + 2 (x - 1) - 12 (x - 1)^2 + 8 (x - 1)^3 + \alpha_3 (x) (x - 1)^3.
\]

Observe that

\[
2 + 2 (x - 1) - 12 (x - 1)^2 + 8 (x - 1)^3 = 8x^3 - 36x^2 + 50x - 20
\]

which shows us that in fact \( \alpha_3 (x) = 0 \) for all \( x \in (\infty, \infty) \). This is not too surprising because the function \( f \) is itself a polynomial of degree three so it is its own best approximation by a polynomial of degree three.

Example 5 Let us approximate the function \( f(x) = e^x \) with a polynomial of degree four near \( x_0 = 0 \). Since \( f^{(n)} (0) = 1 \) for all \( n \geq 1 \), we have

\[
f(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \alpha_4 (x) x \quad \text{for all } x \in (\infty, \infty)
\]

where \( \alpha_4 \) is a function with

\[
\lim_{x \to 0} \alpha_4 (x) = \alpha_4 (0) = 0.
\]

The graphs of \( f \) and the polynomial \( Q(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \) are shown in the figure below.
Exercise 6 Find (explicitly) the function $\alpha_4$ of Example 5 and show by direct computation that $\lim_{x \to 0} \alpha_4 (x) = 0$.

Exercise 7 Find a polynomial of degree 4 that well-approximates $f(x) = \cos x$ near $x_0 = 0$. Graph $f$ together with this polynomial near $x_0 = 0$.

3 Taylor’s Theorem

If $f : (a, b) \to R$ is $n$ times differentiable at all points $x \in (a, b)$ and $n + 1$ times differentiable at $x_0 \in (a, b)$, then

$$f(x) = P_{n+1}(x) + \alpha_{n+1}(x)(x - x_0)^{n+1}$$

for all $x \in (a, b)$ where $P_{n+1}$ is the polynomial function

$$P_{n+1}(x) = f(x_0) + \sum_{j=1}^{n+1} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$$

and $\alpha_{n+1}$ is a function satisfying $\lim_{x \to x_0} \alpha_{n+1} (x) = \alpha_{n+1} (x_0) = 0$. If we define $R_{n+1} : (a, b) \to R$ by

$$R_{n+1}(x) = \alpha_{n+1}(x)(x - x_0)^{n+1},$$

then we can write

$$f(x) = P_{n+1}(x) + R_{n+1}(x).$$
For each integer \( n \geq 1 \), the function \( P_n \) is called the \( nth \) Taylor polynomial of \( f \) at \( x_0 \) and the function \( R_n \) is called the \( nth \) remainder function of \( f \) at \( x_0 \).

Since \( \lim_{x \to x_0} R_{n+1} (x) / (x - x_0)^{n+1} = 0 \), the values of \( R_{n+1} (x) \) approach 0 very rapidly as \( x \to x_0 \). Thus, it seems that the values of \( f(x) \) and \( P_{n+1} (x) \) should be very close when \( x \) is close to \( x_0 \). What do we mean by “close”? In order to be more precise about this, we need to have a better understanding of \( R_{n+1} (x) \). The theorem that gives us this understanding is Taylor’s Theorem named after the English mathematician Brook Taylor (1685-1731) but also discovered independently by Johann Bernoulli.

**Theorem 8 (Taylor’s Theorem)** Suppose that \( f : (a, b) \longrightarrow R \) is \( n+1 \) times differentiable at all points \( x \in (a, b) \) and suppose that \( x_0 \in (a, b) \). Then for any \( x \in (a, b) \) with \( x \neq x_0 \), there exists a point \( c \) that lies between \( x_0 \) and \( x \) such that

\[
f(x) = P_n (x) + \frac{f^{(n+1)} (c)}{(n+1)!} (x - x_0)^{n+1}.
\]

**Proof.** Let \( x \in (x_0, b) \). (The proof that follows is similar if we assume that \( x \in (a, x_0) \).) Since, in this proof, we will be working on the interval \([x_0, x]\), we will use \( t \) as an independent variable with \( x_0 \leq t \leq x \).

By Theorem 3, we have

\[
f(t) = P_{n+1} (t) + R_{n+1} (t) \text{ for all } t \in [x_0, x]
\]

Since we are assuming that \( f \) is \( n+1 \) times differentiable at all points \( t \in (a, b) \), then \( R_{n+1} \) is also \( n+1 \) times differentiable at all points \( t \in (a, b) \) (because \( R_{n+1} = f - P_{n+1} \)) and

\[
R_{n+1}^{(1)} (t) = f^{(1)} (t) - f^{(1)} (x_0) - \sum_{j=2}^{n+1} \frac{f^{(j)} (x_0)}{(j-1)!} (t - x_0)^{j-1}
\]

\[
R_{n+1}^{(2)} (t) = f^{(2)} (t) - f^{(2)} (x_0) - \sum_{j=3}^{n+1} \frac{f^{(j)} (x_0)}{(j-2)!} (t - x_0)^{j-2}
\]

\[
\vdots
\]

\[
R_{n+1}^{(n)} (t) = f^{(n)} (t) - f^{(n)} (x_0) - f^{(n+1)} (x_0) (t - x_0)
\]

\[
P_{n+1}^{(n+1)} (t) = f^{(n+1)} (t) - f^{(n+1)} (x_0)
\]

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We now define the function $G : (a, b) \rightarrow R$ by

$$G(t) = (t - x_0)^{n+1}.$$ 

This derivatives of $G$ of orders 1 through $n + 1$ are

$$G^{(1)}(t) = \frac{(n + 1)!}{n!} (t - x_0)^n$$

$$G^{(2)}(t) = \frac{(n + 1)!}{(n - 1)!} (t - x_0)^{n-1}$$

$$\vdots$$

$$G^{(n)}(t) = (n + 1)! (t - x_0)$$

$$G^{(n+1)}(t) = (n + 1)!$$

We will now apply the Cauchy Mean Value Theorem to the functions $R_{n+1}$ and $G$ on the interval $[x_0, x]$. Both $R_{n+1}$ and $G$ are differentiable at all points of $[x_0, x]$, $G(x_0) \neq G(x)$ (Why?), and $R_{n+1}$ and $G$ are not simultaneously zero at any point of $(x_0, x)$ (Why?). Hence, by the Cauchy Mean Value Theorem, there exists a point $c_1 \in (x_0, x)$ such that

$$\frac{R_{n+1}(x)}{G(x)} = \frac{R_{n+1}(x) - R_{n+1}(x_0)}{G(x) - G(x_0)} = \frac{R_{n+1}^{(1)}(c_1)}{G^{(1)}(c_1)}.$$ 

Now note that $R_{n+1}^{(1)}$ and $G^{(1)}$ are both differentiable at all points of $[x_0, c_1]$, $G^{(1)}(x_0) \neq G^{(1)}(c_1)$, and $R_{n+1}^{(1)}$ and $G^{(1)}$ are never simultaneously zero in $(x_0, c_1)$ so we can again apply the Cauchy Mean Value Theorem (to $R_{n+1}$ and $G^{(1)}$ on $[x_0, c_1]$) to conclude that there exists a point $c_2 \in (x_0, c_1)$ such that

$$\frac{R_{n+1}(x)}{G(x)} = \frac{R_{n+1}^{(1)}(c_1) - R_{n+1}^{(1)}(x_0)}{G^{(1)}(c_1) - G^{(1)}(x_0)} = \frac{R_{n+1}^{(2)}(c_2)}{G^{(2)}(c_2)}.$$ 

Continuing in this fashion (applying the Cauchy Mean Value Theorem a total of $n + 1$ times), we find that there exists a point $c_{n+1} \in (x_0, x)$ such that

$$\frac{R_{n+1}(x)}{G(x)} = \frac{R_{n+1}^{(n+1)}(c_{n+1})}{G^{(n+1)}(c_{n+1})}.$$ 

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This gives us

\[
R_{n+1}(x) = \frac{f^{(n+1)}(c_{n+1})}{(x-x_0)^{n+1}} - \frac{f^{(n+1)}(x_0)}{(n+1)!} (x-x_0)^{n+1}.
\]

or

\[
R_{n+1}(x) = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!} (x-x_0)^{n+1}.
\]

Letting \( c = c_{n+1} \) and substituting into equation (4), we obtain

\[
f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}
\]

and the proof is complete. ■

**Example 9** For \( f(x) = e^x \) and \( x_0 = 0 \), we have for any \( x \neq 0 \),

\[
f(x) = 1 + x + \frac{1}{2} x^2 + \frac{f^{(3)}(c)}{3!} x^3
\]

where \( c \) is some point between 0 and \( x \). Thus,

\[
e^x = 1 + x + \frac{1}{2} x^2 + \frac{e^c}{6} x^3.
\]

Suppose that we wish to get a numerical approximation the value of \( e^{1/2} \) (without using a calculator). We obtain

\[
e^{1/2} = 1 + \frac{1}{2} + \frac{1}{2} \left( \frac{1}{2} \right)^2 + \frac{1}{6} \left( \frac{1}{2} \right)^3 e^c = \frac{78}{48} + \frac{1}{48} e^c
\]

where \( c \) lies between 0 and 1/2.

Since \( 0 < c < 1/2 \) and \( f(x) = e^x \) is a monotone increasing function, we have \( e^0 < e^c < e^{1/2} \). This gives us

\[
\frac{1}{48} e^0 < \frac{1}{48} e^c < \frac{1}{48} e^{1/2}
\]

and

\[
\frac{78}{48} + \frac{1}{48} e^0 < \frac{78}{48} + \frac{1}{48} e^c < \frac{78}{48} + \frac{1}{48} e^{1/2}
\]

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so

\[ \frac{79}{48} < e^{1/2} < \frac{78}{48} + \frac{1}{48} e^{1/2} \]

Using

\[ e^{1/2} < \frac{78}{48} + \frac{1}{48} e^{1/2} , \]

we obtain

\[ e^{1/2} < \frac{78}{47} \]

which gives us an overall estimate of

\[ \frac{79}{48} < e^{1/2} < \frac{78}{47} \]

This estimate is pretty good estimate because \(79/48 \approx 1.6458\) and \(78/47 \approx 1.6596\). If you compute \(e^{1/2}\) on a calculator, you will get 1.648721271.

**Exercise 10** Use \(1 + 1 + \frac{1}{2!}(1)^2 + \frac{1}{3!}(1)^3\) to approximate \(e\) (without using a calculator) and then use Taylor’s Theorem to estimate how good your approximation is. Use a calculator to compute \(e\) and compare.

**Exercise 11** For the function \(f(x) = \cos x\), use

\[ f(0) + \sum_{j=1}^{4} \frac{f^{(j)}(0)}{j!} (x - 0)^j \]

to approximate \(\cos 1\) (without using a calculator) and then use Taylor’s Theorem to estimate how good your approximation is. Use a calculator to compute \(\cos 1\) and compare.

**Exercise 12** Show that given any \(x \in (-\infty, \infty)\) and given any \(\varepsilon > 0\), there exists an integer \(n \geq 1\) such that \(|\cos x - P_n(x)| < \varepsilon\) (where \(P_n\) is the \(n\)th Taylor polynomial of \(f(x) = \cos x\) at \(x_0 = 0\)).

As a specific illustration, find \(n\) such that \(|\cos 10 - P_n(10)| < 0.00001\).

**Exercise 13** For the function \(f : (-\infty, \infty) \rightarrow R\) defined by

\[ f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} , \]

show that \(P_n(x) = 0\) for all \(x \in (-\infty, \infty)\) and for all \(n \geq 1\).