Uniform Continuity

S. F. Ellermeyer

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1 Definition and Examples

Definition 1 For a function $f : D \rightarrow \mathbb{R}$, we say that $f$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x$ and $y \in D$ with $|x - y| < \delta$.

Example 2 Let $D = [0; 4]$. The function $f : D \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

To see that this is so, let $\varepsilon > 0$ be given and let $\delta = \varepsilon/8$. Then, if $x$ and $y$ are any two points in $D$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < (4 + 4)\delta = 8\delta = \varepsilon/8 = \varepsilon.$$

Example 3 Let $D = [0; 1)$. The function $f : D \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

To show that $f$ is not uniformly continuous, we must show that there exists $\varepsilon > 0$ such that given any $\delta > 0$ there exist $x$ and $y$ in $D$ with $|x - y| < \delta$ but $|f(x) - f(y)| > \varepsilon$. (We will show that this is true using $\varepsilon = 2$.)

Let $\varepsilon = 2$ and let $\delta > 0$ be given. Then choose a positive integer $n$ with $\frac{1}{n} < \delta$ and let $x = n + \frac{1}{n}$ and $y = n$. Then $x$ and $y$ are in $D$ and

$$|x - y| = \frac{1}{n} < \delta.$$

Then

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| = \left(2n + \frac{1}{n}\right)\frac{1}{n} = \frac{2n^2 + 1}{n^2} < \delta.$$
but
\[
\begin{align*}
\left| f(x) - f(y) \right| &= \left| x^2 - y^2 \right| \\
&= \left| x + y \right| \left| x - y \right| \\
&= \left( \frac{1}{n} \right) \frac{1}{n} \\
&= 2 + \frac{1}{n^2} > "
\end{align*}
\]

Exercise 4 Let \( a \) and \( b \) be real numbers with \( a < b \) and let \( D = [a; b] \). Show that the function \( f : D \rightarrow \mathbb{R} \) defined by \( f(x) = x^2 \) is uniformly continuous.

Exercise 5 Let \( D = (0; 1] \) and let \( f : D \rightarrow \mathbb{R} \) be the function defined by \( f(x) = \frac{1}{x} \). Show that \( f \) is not uniformly continuous.

2 Relating Continuity and Uniform Continuity

Note that uniform continuity of a function is a property that depends greatly on the domain of the function. For example, the functions in Examples 2 and 3 are both defined by the same rule, \( f(x) = x^2 \), but one is uniformly continuous and the other is not. Also, it is important to observe that a function \( f : D \rightarrow \mathbb{R} \) might be continuous at each point of \( D \) but not be uniformly continuous. However, if \( f : D \rightarrow \mathbb{R} \) is uniformly continuous, then \( f \) must be continuous at each point of \( D \). This result is given in the following theorem.

Theorem 6 If \( f : D \rightarrow \mathbb{R} \) is uniformly continuous, then \( f \) is continuous at each point of \( D \).

Proof. Let \( x_0 \) be a point in \( D \) and let \( " > 0 \) be given. Since \( f \) is uniformly continuous, there exists \( \varepsilon > 0 \) such that if \( x \) and \( y \) are any two points in \( D \) with \( \left| x - y \right| < \varepsilon \), then \( \left| f(x) - f(y) \right| < " \). In particular, since \( x_0 \in D \), then if \( x \) is any point in \( D \) with \( \left| x - x_0 \right| < \varepsilon \), then \( \left| f(x) - f(x_0) \right| < " \). This shows that \( f \) is continuous at \( x_0 \). Since \( x_0 \) was chosen arbitrarily form \( D \), \( f \) is continuous at each point of \( D \). ■
As was remarked earlier, the converse of Theorem 6 is not true in general. However, if \( D = [a; b] \) (a closed interval), then the converse of Theorem 6 is true.

**Theorem 7** Let \( a \) and \( b \) be real numbers with \( a < b \) and let \( f : [a; b] \rightarrow \mathbb{R} \). If \( f \) is continuous at each point of \([a; b]\), then \( f \) is uniformly continuous.

**Proof.** Suppose that \( f \) is not uniformly continuous. Then there exists \( \varepsilon > 0 \) such that for any \( \delta > 0 \) there exist points \( x \) and \( y \) in \([a; b]\) with \( jx - yj < \delta \) but \( jf(x) - f(y)j > \varepsilon \). In particular for each integer \( n \geq 1 \), there exist points \( x_n \) and \( y_n \) in \([a; b]\) with \( jx_n - y_nj < \frac{1}{n} \) but \( jf(x_n) - f(y_n)j \geq \varepsilon \). Clearly, \( \lim_{n \to \infty} jx_n - y_nj = 0 \).

Since the sequence \( x_n \) is bounded, it must have a cluster point \( c \) in \([a; b]\). This means that there exists a subsequence, \( x_{n_k} \), of \( x_n \) such that \( x_{n_k} \to c \). Since \( jx_{n_k} - y_{n_k}j \to 0 \), it must also be true that \( y_{n_k} \to c \). However, since \( jf(x_{n_k}) - f(y_{n_k})j \to 0 \) for all \( k \geq 1 \), then either \( f(x_{n_k}) \to f(c) \) or \( f(y_{n_k}) \to f(c) \) and we conclude that \( f \) is not continuous at \( c \).

We have shown that if \( f \) is not uniformly continuous, then there exists a point, \( c \in [a; b] \), at which \( f \) is not continuous. This proves the theorem. \( \blacksquare \)

**Exercise 8** Example 3 provides an example of a function \( f : [0; 1) \to \mathbb{R} \) that is not uniformly continuous. Give an example of a function \( f : [0; 1) \to \mathbb{R} \) that is uniformly continuous (thus showing that unboundedness of the domain of \( f \) does not rule out the possibility that \( f \) is uniformly continuous).

**Exercise 9** Give an example of a set \( D \), and a function \( f : D \to \mathbb{R} \) that is bounded on \( D \) and continuous at each point of \( D \), but not uniformly continuous (thus showing that boundedness and continuity are not sufficient for uniform continuity).

### 3 Lipschitz Continuity

Theorem 6 states that uniform continuity of \( f : D \to \mathbb{R} \) guarantees continuity of \( f \) at each point in \( D \). Here, we discuss a type of continuity, Lipschitz continuity, that is stronger than uniform continuity.
Definition 10 A function \( f: D \rightarrow \mathbb{R} \) is said to be Lipschitzian (or to satisfy a Lipschitz condition or to be Lipschitz continuous) if there exists a number \( K > 0 \) such that \( |f(x) - f(y)| \leq K |x - y| \) for all \( x \) and \( y \in D \).

Example 11 The function \( f: [0; 4] \rightarrow \mathbb{R} \) defined by \( f(x) = x^2 \) is Lipschitzian because for any \( x \) and \( y \in [0; 4] \), we have
\[
|f(x) - f(y)| = x^2 - y^2 = x + y |x - y| = (|x| + |y|) |x - y| + 8x |y| < K |x - y|,
\]
for \( K = 8 \).

Exercise 12 Let \( a \) and \( b \) be real numbers with \( a < b \) and let \( f: [a; b] \rightarrow \mathbb{R} \) be the function defined by \( f(x) = x^2 \). Show that \( f \) is Lipschitzian.

Exercise 13 Let \( f: [0; 1) \rightarrow \mathbb{R} \) be the function defined by \( f(x) = x/(x^2 + 1) \). Show that \( f \) is Lipschitzian.

Theorem 14 If \( f: D \rightarrow \mathbb{R} \) is Lipschitzian, then \( f \) is uniformly continuous.

Proof. Since \( f \) is Lipschitzian, there exists \( K > 0 \) such that \( |f(x) - f(y)| \leq K |x - y| \) for all \( x \) and \( y \in D \). Thus, if we are given \( \varepsilon > 0 \) and we choose \( \delta = \varepsilon/K \), then for any \( x \) and \( y \) in \( D \) with \( |x - y| < \delta \) we have
\[
|f(x) - f(y)| < K \delta = \varepsilon.
\]

Exercise 15 Give an example of a set \( D \) and a function \( f: D \rightarrow \mathbb{R} \) that is uniformly continuous but not Lipschitzian.

4 The Sequential Approach to Uniform Continuity

Recall that if \( f: D \rightarrow \mathbb{R} \) and \( x_0 \in D \), then \( f \) is continuous at \( x_0 \) if and only if for every sequence of points, \( x_n \), in \( D \) with \( x_n \rightarrow x_0 \), we also have \( f(x_n) \rightarrow f(x_0) \). An equivalent way to state this is that \( f \) is continuous at \( x_0 \) if and only if for every sequence of points, \( x_n \), in \( D \) with \( |x_n - x_0| \rightarrow 0 \), we also have \( |f(x_n) - f(x_0)| \rightarrow 0 \).

There is a similar characterization of uniform continuity in terms of sequences which is given in the following theorem.
Theorem 16 Let \( f : D \to \mathbb{R} \). Then \( f \) is uniformly continuous if and only if for every pair of sequences, \( x_n \) and \( y_n \), of points in \( D \) with \( |x_n - y_n| \to 0 \), we also have \( |f(x_n) - f(y_n)| \to 0 \).

Proof. Suppose that \( f \) is uniformly continuous and let \( x_n \) and \( y_n \) be sequences of points in \( D \) with \( |x_n - y_n| \to 0 \). We must show that it is also true that \( |f(x_n) - f(y_n)| \to 0 \). To do this, we let \( \varepsilon > 0 \) be given and we must find an integer \( m \) such that

\[ |f(x_n) - f(y_n)| < \varepsilon \]

for all \( n \geq m \).

Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon \) for all points \( x \) and \( y \) in \( D \) with \( |x - y| < \delta \). Also, since \( jx_n - y_n| \to 0 \), there exists an integer \( m \) such that \( |x_n - y_n| < \delta \) for all \( n \geq m \). Hence if we take any \( n \geq m \), then \( x_n \) and \( y_n \) are points in \( D \) with \( |x_n - y_n| < \delta \) and it follows that \( |f(x_n) - f(y_n)| < \varepsilon \).

We now prove the converse: Suppose that \( f \) is not uniformly continuous. Then there exists \( \varepsilon > 0 \) such that for any \( \delta > 0 \) there exist points \( x \) and \( y \) in \( D \) with \( |x - y| < \delta \) but \( |f(x) - f(y)| \geq \varepsilon \). In particular, for each integer \( n \geq 1 \), there exist points \( x_n \) and \( y_n \) in \( D \) with \( |x_n - y_n| < 1/n \) but \( |f(x_n) - f(y_n)| \geq \varepsilon \). Clearly, \( |x_n - y_n| \to 0 \) but \( |f(x_n) - f(y_n)| \not\to 0 \).

Example 17 Let us use Theorem 16 to show that \( f : [0; 1) \to \mathbb{R} \) defined by \( f(x) = x^2 \) is not uniformly continuous. (The reader will notice that this is essentially a reworking of Example 3.)

Let \( x_n = n \cdot \frac{1}{n} \) and let \( y_n = n \). Then \( x_n \) and \( y_n \) are sequences of points in \([0; 1) \) with \( |x_n - y_n| = \frac{1}{n} \to 0 \). However,

\[ |f(x_n) - f(y_n)| = n + \frac{1}{n} \geq 1 \]

so \( f \) is not uniformly continuous.

Exercise 18 Let \( f : (0; 1) \to \mathbb{R} \) be defined by \( f(x) = 1/x \). Use Theorem 16 to show that \( f \) is not uniformly continuous.