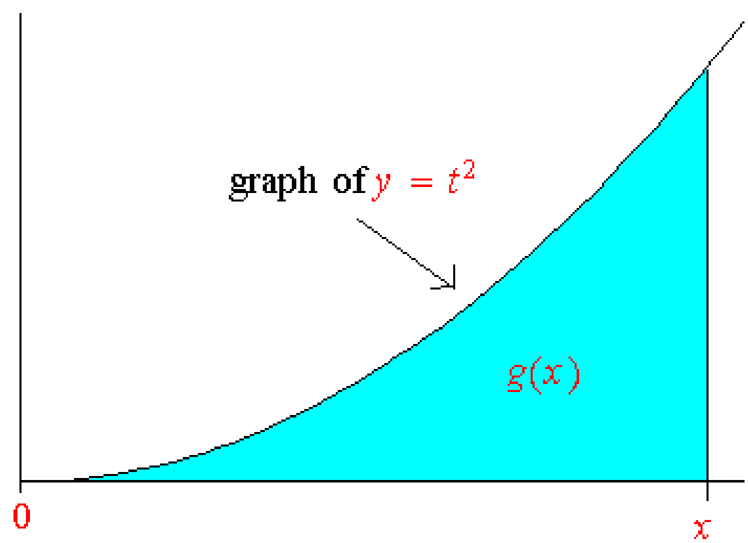
The Fundamental Theorem of Calculus

Consider the function

$$g(x) = \int_0^x t^2\,dt.$$

For each $x > 0$, $g(x)$ is the area determined by the graph of the curve $y = t^2$ over the interval $[0, x]$. (See the figure below.)



Using the Evaluation Theorem and the fact that the function $F(t) = \frac{1}{3}t^3$ is an antiderivative of the function $f(t) = t^2$, we can obtain a more explicit formula for $g(x)$:

$$\begin{aligned} g(x) &= \int_0^x t^2\,dt \\ &= \frac{1}{3}(x)^3 - \frac{1}{3}(0)^3 \\ &= \frac{1}{3}x^3. \end{aligned}$$

Thus, g is simply the function

$$g(x) = \frac{1}{3}x^3,$$

which means that

$$g'(x) = x^2.$$

To summarize, we have found that

$$\frac{d}{dx}\left(\int_0^x t^2\,dt\right) = x^2.$$

This is not a coincidence! In fact, The Fundamental Theorem of Calculus tells us that this will always occur. Specifically, if f is any continuous function and g is a function defined by

$$g(x) = \int_a^x f(t)\,dt$$

(where a can be any number such that the interval $[a,x]$ is contained in the domain of f), then it will always be true that

$$g'(x) = f(x),$$

or, stated in Leibniz notation,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Let us look at one more specific example: Suppose that g is the function

$$g(x) = \int_0^x \cos(t)\,dt.$$

Then, since $F(t) = \sin(t)$ is an antiderivative of $f(t) = \cos(t)$, we can use the Evaluation Theorem to write g more explicitly as

$$\begin{aligned} g(x) &= \int_5^x \cos(t)\,dt \\ &= \sin(x) - \sin(5). \end{aligned}$$

From this, we see that

$$g'(x) = \cos(x).$$

In Leibniz notation:

$$\frac{d}{dx}\left(\int_5^x \cos(t)\,dt\right) = \cos(x).$$

The phenomenon illustrated in the previous two examples is what is called the First Part of the Fundamental Theorem of Calculus. The Second Part of the Fundamental Theorem of Calculus is the Evaluation Theorem that we have already stated. Here is a formal statement of the Fundamental Theorem of Calculus:

Theorem (The Fundamental Theorem of Calculus)

Part* 1*: *If f is a continuous function and a is a number in the domain of f and we define the function g by*

$$g(x) = \int_a^x f(t)\,dt,$$

then

$$g'(x) = f(x).$$

In other words,

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Part* 2*: *If f is a continuous function and F is an antiderivative of f on the interval* $[a,b]$, *then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example *Find the derivative of the function*

$$g(x) = \int_0^x \sqrt{1+t^2}\, dt.$$

Example *Find the derivative of the function*

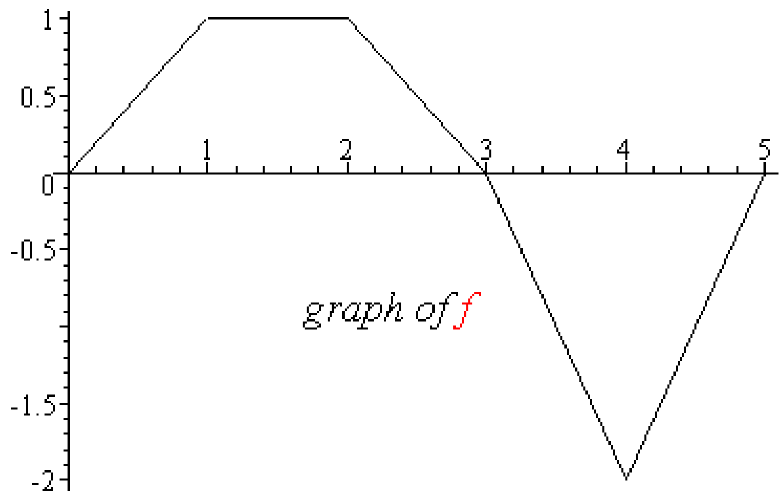
$$g(x) = \int_1^{cos(x)} t^2\, dt$$

in two different ways:

a) by first using the Evaluation Theorem (which is Part 2 *of the Fundamental Theorem of Calculus) to obtain an explicit expression for* $g(x)$ *and then computing* $g'(x)$

b) by using Part 1 *of the Fundamental Theorem of Calculus together with the Chain Rule.*

Example *Let f be the function whose graph is shown below.*



Also, let g be the function

$$g(x) = \int_0^x f(t)\,dt.$$

Use the graph of f to compute the values $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, *and* $g(5)$. *Then make a rough sketch of the graph of g over the entire interval* $[0,5]$.

According to the Fundamental Theorem of Calculus (Part 1), $g'(x) = f(x)$. *Do you observe this when you compare the graphs of f and g?*