Arc Length
Distance Between Two Points

If \((x_1, y_1)\) and \((x_2, y_2)\) are any two points in the plane, then the distance between these points is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]
An approximation of the length of the parametric curve, $C$, defined by $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is

$$\sum_{i=1}^{n} \sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2}$$
If \( f \) and \( g \) are both differentiable functions of \( t \), then we can apply the Mean Value Theorem to obtain that the approximate arc length is

\[
\sum_{i=1}^{n} \sqrt{(f'(t_i^*))^2 + (g'(t_i^#))^2} \Delta t
\]

where \( t_i^* \) and \( t_i^# \) are points between \( t_{i-1} \) and \( t_i \). Taking the limit as \( n \to \infty \) (and assuming that \( f' \) and \( g' \) are both continuous functions), we obtain that the actual arc length is

\[
\int_{a}^{b} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt.
\]
Example 1

Show that the arc length (circumference) of a circle of radius $r$ is $2\pi r$. 
Example 2

Write down a definite integral that gives the arc length of the ellipse

\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1.
\]
Example 3

Find the arc length of the piece of the parabola $y = x^2$ that extends from the point $(0, 0)$ to the point $(1, 1)$. 
Special Case: $y = g(x)$

For the graph of a continuously differentiable function, $y = g(x), a \leq x \leq b$, we can parameterize this curve as $x = t, y = g(t), a \leq t \leq b$. Thus $dx/dt = 1$, $dy/dt = g'(t)$ and the arc length formula becomes

$$\text{Arc Length} = \int_{a}^{b} \sqrt{1 + (g'(x))^2} \, dx.$$
Example 4

Find the arc length of the piece of the parabola $y = x^2$ that extends from the point $(0, 0)$ to the point $(1, 1)$ by using the formula

$$\text{Arc Length} = \int_{a}^{b} \sqrt{1 + (g'(x))^2} \, dx.$$
Example 5

Find the arc length of one arch of the cycloid

\[ x = 2(t - \sin(t)), \quad y = 2(1 - \cos(t)). \]
Homework

In the textbook, Section 6.3 (page 465), do problems 1-10 (all), 14, and 24.