Definition of the Inverse Secant Function

From trigonometry, we know that the secant function is defined by

\[ \sec(x) = \frac{1}{\cos(x)}. \]

Since \( \cos(x) = 0 \) at \( x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \text{ etc.} \) (all odd multiples of \( \pi/2 \)), the secant function is not defined at these values of \( x \) (since division by zero is undefined). Also, since the range of the cosine function is \( [-1, 1] \), the range of the secant function is \( (-\infty, -1] \cup [1, \infty) \). A portion of the graph of the secant function is shown below. The graph has vertical asymptotes at all odd multiples of \( \pi/2 \).

[Graph of \( y = \sec(x) \)]

If we restrict the domain of the secant function to \([0, \pi/2) \cup (\pi/2, \pi]\) (as shown in the graph below), then we have a one–to–one function which thus has an inverse.
The inverse secant function (denoted by \( \text{arcsec} \)) is defined using the restricted domain on the secant function described above. Thus the inverse secant function has domain \((-\infty, -1] \cup [1, \infty)\) and range \([0, \pi/2) \cup (\pi/2, \pi]\). For each \( x \) in the set \((-\infty, -1] \cup [1, \infty)\) we define
\[
y = \text{arcsec}(x)
\]
to be the unique value of \( y \) in the set \([0, \pi/2) \cup (\pi/2, \pi]\) for which
\[
\sec(y) = x.
\]
A graph of the inverse secant function is shown below.

To make sure that we understand the definition of the \( \text{arcsec} \) function, let us consider a couple of examples.

**Example**  
*What is the value of \( \text{arcsec}(1) \)?*
To answer this we must realize that
\[ y = \text{arcsec}(1) \]

means that
\[ \sec(y) = 1 \]
and that \( y \) is a number in \([0, \pi/2) \cup (\pi/2, \pi]\). Based on our knowledge of trigonometric functions we know that \( \cos(0) = 1 \) and hence
\[ \sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1. \]

Furthermore we see that 0 is a number in the set \([0, \pi/2) \cup (\pi/2, \pi]\). Therefore \( \text{arcsec}(1) = 0 \).

Why isn’t it true that \( \text{arcsec}(1) = 2\pi \)? After all it is true that
\[ \sec(2\pi) = \frac{1}{\cos(2\pi)} = \frac{1}{1} = 1. \]

However note that the number 2\( \pi \) is not in the set \([0, \pi/2) \cup (\pi/2, \pi]\). Thus \( \text{arcsec}(1) \neq 2\pi \) even though \( \sec(2\pi) = 1 \).

Example

What is the value of \( \text{arcsec}(\sqrt{2}) \)?

To answer this we must realize that
\[ y = \text{arcsec}(\sqrt{2}) \]

means that
\[ \sec(y) = \sqrt{2} \]
and that \( y \) is a number in the set \([0, \pi/2) \cup (\pi/2, \pi]\). Based on our knowledge of trigonometric functions we know that \( \cos(3\pi/4) = -\sqrt{2}/2 \) and hence
\[ \sec(3\pi/4) = \frac{1}{\cos(3\pi/4)} = \frac{1}{-\sqrt{2}/2} = -\sqrt{2}. \]

Furthermore we see that \( 3\pi/4 \) is a number in the set \([0, \pi/2) \cup (\pi/2, \pi]\). Therefore \( \text{arcsec}(\sqrt{2}) = 3\pi/4 \).

Why isn’t it true that \( \text{arcsec}(\sqrt{2}) = -3\pi/4 \)? After all it is true that
\[ \sec(-3\pi/4) = \frac{1}{\cos(-3\pi/4)} = \frac{1}{-\sqrt{2}/2} = -\sqrt{2}. \]

However note that the number \( -3\pi/4 \) is not in the set \([0, \pi/2) \cup (\pi/2, \pi]\). Thus \( \text{arcsec}(\sqrt{2}) \neq -3\pi/4 \) even though \( \sec(-3\pi/4) = -\sqrt{2} \). In fact, notice that the value of \( \text{arcsec}(x) \) is never negative for any value of \( x \). (Look at the graph of the \( \text{arcsec} \) function shown above.)

Exercise

Determine the values of the following. (Each can be done without a calculator – just using knowledge of the unit circle and the definition of \( \text{arcsec} \)).

1. \( \text{arcsec}(2\sqrt{3}/3) \)
2. \( \text{arcsec} \left( \frac{\sqrt{2}}{2} \right) \)
3. \( \text{arcsec}(2) \)
4. \( \text{arcsec} \left( -2, \frac{\sqrt{3}}{3} \right) \)
5. \( \text{arcsec}(-2) \)
6. \( \text{arcsec}(-1) \)

**The Derivative of \( \text{arcsec} \)**

We now compute the derivative of \( y = \text{arcsec}(x) \). Rewriting \( y = \text{arcsec}(x) \) as \( \sec(y) = x \) and using implicit differentiation gives

\[
\frac{d}{dx} (\sec(y)) = \frac{d}{dx} (x)
\]

or

\[
\sec(y) \tan(y) \frac{dy}{dx} = 1
\]

or

\[
\frac{dy}{dx} = \frac{1}{\sec(y) \tan(y)}.
\]

Using the trigonometric identity

\[
\sec^2(y) = 1 + \tan^2(y)
\]

we obtain

\[
\tan(y) = \pm \sqrt{\sec^2(y) - 1}.
\]

If \( x \) is in the interval \((1, \infty)\), then \( y \) is in the interval \((0, \pi/2)\) (see the graph of the \( \text{arcsec} \) function) meaning that \( \tan(y) > 0 \) and we choose the “+” sign on the above square root. This gives us

\[
\frac{dy}{dx} = \frac{1}{x \sqrt{x^2 - 1}}.
\]

However, if \( x \) is in the interval \((-\infty, -1)\), then \( y \) is in the interval \((\pi/2, \pi)\) meaning that \( \tan(y) < 0 \) and we choose the “−” sign on the square root and this gives us

\[
\frac{dy}{dx} = \frac{1}{-x \sqrt{x^2 - 1}}.
\]

At \( x = 1 \) and \( x = -1 \), the \( \text{arcsec} \) function is not differentiable. Its graph has a vertical (infinite slope) tangent line at these points.

In conclusion, the derivative of the inverse secant function is
\[
\frac{d}{dx} \text{arcsec}(x) = \begin{cases} 
\frac{1}{-\sqrt{x^2 - 1}} & \text{if } -\infty < x < -1 \\
\frac{1}{x \sqrt{x^2 - 1}} & \text{if } 1 < x < \infty 
\end{cases}
\]

Integrals Involving arcsec

From the above differentiation formula, we can see that the value of

\[
\int \frac{1}{x \sqrt{x^2 - 1}} \, dx
\]

depends on what interval we are working with. If we are working on some interval that is a subset of \((1, \infty)\), then

\[
\int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \text{arcsec}(x) + C.
\]

However if we are working on some interval that is a subset of \((-\infty, -1)\), then

\[
\int \frac{1}{x \sqrt{x^2 - 1}} \, dx = -\text{arcsec}(x) + C
\]

because on such an interval we have

\[
\frac{d}{dx}(-\text{arcsec}(x)) = -\frac{d}{dx}(\text{arcsec}(x)) = -\left(\frac{1}{-x \sqrt{x^2 - 1}}\right) = \frac{1}{x \sqrt{x^2 - 1}}.
\]

For example, the interval \([\sqrt{2}, 2]\) is a subset of \((1, \infty)\) and thus

\[
\int_{\sqrt{2}}^{2} \frac{1}{x \sqrt{x^2 - 1}} \, dx = \text{arcsec}(x) \bigg|_{\sqrt{2}}^{2} = \text{arcsec}(2) - \text{arcsec}(\sqrt{2}) = \frac{\pi}{12}.
\]

However, the interval \([-\sqrt{2}, -2]\) is a subset of \((-\infty, -1)\) and hence

\[
\int_{-\sqrt{2}}^{-2} \frac{1}{x \sqrt{x^2 - 1}} \, dx = -\text{arcsec}(x) \bigg|_{-2}^{-\sqrt{2}} = -\text{arcsec}(-\sqrt{2}) - (-\text{arcsec}(-2)) = -\frac{\pi}{12}.
\]