1. Let $C$ be the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$. Let $F$ be the vector field

$$F(x, y) = -\sin(y)i + x\cos(y)j.$$ 

The vector field and curve are pictured together below.

Compute the clockwise circulation of $F$ around $C$. It is up to you whether you would rather do this directly or use Green’s Theorem.

Solution 1: The circulation is

$$\int_C (Mdx + Ndy).$$

The functions $M$ and $N$ are

$$M = -\sin(y)$$

$$N = x\cos(y).$$

If we wish to compute the circulation directly, we need to parameterize the four line segments that make up $C$. The segment, $C_1$, on the bottom can be parameterized by
\[x = x \quad y = 0\]
\[dx = dx \quad dy = 0\]

and we obtain
\[
\int_{C_1} (M \, dx + N \, dy) = \int_0^{\pi/2} -\sin(0) \, dx = 0.
\]

The segment, \(C_2\), on the right can be parameterized by
\[x = \pi/2 \quad y = y\]
\[dx = 0 \quad dy = dy\]

and we obtain
\[
\int_{C_2} (M \, dx + N \, dy) = \int_0^{\pi/2} \frac{\pi}{2} \cos(y) \, dy = \frac{\pi}{2}.
\]

The segment, \(C_3\), on the top can be parameterized by
\[x = x \quad y = \pi/2\]
\[dx = dx \quad dy = 0\]

and we obtain
\[
\int_{C_3} (M \, dx + N \, dy) = \int_{\pi/2}^{0} -\sin\left(\frac{\pi}{2}\right) \, dx = \frac{\pi}{2}.
\]

The final segment, \(C_4\), can be parameterized by
\[x = 0 \quad y = y\]
\[dx = 0 \quad dy = dy\]

and we obtain
\[
\int_{C_4} (M \, dx + N \, dy) = \int_{C_4} N \, dy = \int_{\pi/2}^{0} \cos(y) \, dy = 0.
\]

The circulation around \(C\) is thus
\[0 + \frac{\pi}{2} + \frac{\pi}{2} + 0 = \pi.
\]

**Solution 2:** By Green’s Theorem, the circulation is
\[
\int_C (M \, dx + N \, dy) = \iint_D (N_x - M_y) \, dA
\]
\[
= \int_0^{\pi/2} \int_0^{\pi/2} (\cos(y) + \cos(y)) \, dy \, dx
\]
\[
= 2 \int_0^{\pi/2} \int_0^{\pi/2} \cos(y) \, dy \, dx
\]
\[
= 2 \int_0^{\pi/2} 1 \, dx
\]
\[
= \pi.
\]
2. Let $C$ be the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = -\sin(y)\mathbf{i} + x\cos(y)\mathbf{j}.$$ 

The vector field and curve are pictured together in problem 1. Compute the **outward flux** of $\mathbf{F}$ across $C$. It is up to you whether you would rather do this directly or use Green’s Theorem.

**Solution 1:** The outward flux is

$$\int_C (M\,dy - N\,dx).$$

The functions $M$ and $N$ are

$$M = -\sin(y)$$

$$N = x\cos(y).$$

If we wish to compute the flux directly, we need to parameterize the four line segments that make up $C$. The segment, $C_1$, on the bottom can be parameterized by

$$x = x \quad y = 0$$

$$dx = dx \quad dy = 0$$

and we obtain

$$\int_{C_1} (M\,dy - N\,dx) = \int_0^{\pi/2} -x\cos(0)\,dx = -\frac{\pi^2}{8}.$$

The segment, $C_2$, on the right can be parameterized by

$$x = \pi/2 \quad y = y$$

$$dx = 0 \quad dy = dy$$

and we obtain

$$\int_{C_2} (M\,dy - N\,dx) = \int_0^{\pi/2} -\sin(y)\,dy = -1.$$

The segment, $C_3$, on the top can be parameterized by

$$x = x \quad y = \pi/2$$

$$dx = dx \quad dy = 0$$

and we obtain

$$\int_{C_3} (M\,dy - N\,dx) = \int_0^{\pi/2} -x\cos\left(\frac{\pi}{2}\right)\,dx = 0.$$

The final segment, $C_4$, can be parameterized by

$$x = 0 \quad y = y$$

$$dx = 0 \quad dy = dy$$

and we obtain
\[
\int_{C_4} (M \, dy - N \, dx) = \int_{C_4} M \, dy = \int_{\pi/2}^{0} -\sin(y) \, dy = 1.
\]

The flux across \( C \) is thus

\[- \frac{\pi^2}{8} - 1 + 0 + 1 = -\frac{\pi^2}{8}.\]

**Solution 2:** By Green’s Theorem, the flux is

\[
\int_{C} (M \, dy - N \, dx) = \iint_{D} (M_x + N_y) \, dA
\]

\[
= \int_{0}^{\pi/2} \int_{0}^{\pi/2} (0 - x \sin(y)) \, dy \, dx
\]

\[
= -\int_{0}^{\pi/2} \int_{0}^{\pi/2} x \sin(y) \, dy \, dx
\]

\[
= \left( \int_{0}^{\pi/2} x \, dx \right) \left( \int_{0}^{\pi/2} -\sin(y) \, dy \right)
\]

\[
= \frac{1}{2} \left( \frac{\pi}{2} \right)^2 (0 - 1)
\]

\[
= -\frac{\pi^2}{8}.
\]

3. The band of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies between the planes \( z = 1 \) and \( z = \sqrt{2} \) is pictured below.

![Diagram of the sphere](image)

Write parametric equations for the portion of the sphere pictured. (Note that it is necessary to include the ranges of parameter values that describe the given portion of the sphere.)

**Solution:** We know that the sphere (which has radius 2) can be parameterized as

\[
x = 2 \cos(\theta) \sin(\phi)
\]

\[
y = 2 \sin(\theta) \sin(\phi)
\]

\[
z = 2 \cos(\phi).
\]
Since we want the part in the whole band, it is easy to see that we must take

\[ 0 \leq \theta \leq 2\pi. \]

On the sphere we have \( x^2 + y^2 + z^2 = 4 \). The sphere intersects the plane \( z = 1 \) when \( 2\cos(\phi) = 1 \) or \( \cos(\phi) = 1/2 \) and thus \( \phi = \pi/3 \). The sphere intersects the plane \( z = \sqrt{2} \) when \( 2\cos(\phi) = \sqrt{2} \) or \( \cos(\phi) = \sqrt{2}/2 \) and thus \( \phi = \pi/4 \). Thus, to generate the band, we restrict \( \phi \) such that

\[ \frac{\pi}{4} \leq \phi \leq \frac{\pi}{3}. \]

4. Compute the surface area of the band of the sphere given in the previous problem. You may use the fact that for a sphere of radius \( a \), parameterized using spherical coordinates, it is known that \( |r_\theta \times r_\phi| = a^2 \sin(\phi) \).

**Solution:** Since the sphere has radius \( a = 2 \), the surface area is

\[ \int \int_D 4 \sin(\phi) \, dA \]

where

\[ D = \{ (\theta, \phi) \mid 0 \leq \theta \leq 2\pi \text{ and } \pi/4 \leq \phi \leq \pi/3 \}. \]

(This is known by the work done in the previous problem.) The surface area is thus

\[ \int_0^{2\pi} \int_{\pi/4}^{\pi/3} 4 \sin(\phi) \, d\phi \, d\theta = 4 \int_0^{2\pi} -\cos(\phi) \bigg|_{\phi=\pi/3}^{\phi=\pi/4} \, d\theta \]

\[ = 4 \int_0^{2\pi} \left( -\frac{1}{2} - \left( -\frac{\sqrt{2}}{2} \right) \right) \, d\theta \]

\[ = 2(\sqrt{2} - 1) \int_0^{2\pi} 1 \, d\theta \]

\[ = 4(\sqrt{2} - 1)\pi. \]

5. Let \( S \) be the cylindrical surface (pictured)

\[ y^2 + z^2 = 4 \]

\[ z \geq 0 \]

\[ 1 \leq x \leq 4. \]
Evaluate \[ \iint_S z^2 \, d\sigma. \]

**Solution:** The surface \( S \) is
\[
\mathbf{r}(x, \theta) = xi + 2 \cos(\theta)j + 2 \sin(\theta)k
\]
\[
1 \leq x \leq 4
\]
\[
0 \leq \theta \leq \pi.
\]
Thus
\[
\mathbf{r}_x = i
\]
\[
\mathbf{r}_\theta = -2 \sin(\theta)j + 2 \cos(\theta)k
\]
\[
\mathbf{r}_x \times \mathbf{r}_\theta = -2 \sin(\theta)k - 2 \cos(\theta)j
\]
\[
|\mathbf{r}_x \times \mathbf{r}_\theta| = 2.
\]
We thus obtain
\[
\iint_S z^2 \, d\sigma = \iint_D z^2 |\mathbf{r}_x \times \mathbf{r}_\theta| \, dA
\]
\[
= \iint_D 2z^2 \, dA
\]
\[
= 2 \int_1^4 \int_0^\pi 4 \sin^2(\theta) \, d\theta \, dx
\]
\[
= 4 \int_1^4 \int_0^\pi (1 - \cos(2\theta)) \, d\theta \, dx
\]
\[
= 12\pi.
\]

6. Let \( S \) be the cylindrical surface
\[ y^2 + z^2 = 4 \]
\[ z \geq 0 \]
\[ 1 \leq x \leq 4 \]

and let \( \mathbf{F} \) be the vector field
\[ \mathbf{F}(x,y,z) = x \mathbf{k}. \]

The vector field and surface are pictured together below.

Find the flux of \( \mathbf{F} \) in the upward direction across \( S \).

**Solution 1:** The surface \( S \) is
\[ \mathbf{r}(x,\theta) = xi + 2 \cos(\theta)j + 2 \sin(\theta)k \]
\[ 1 \leq x \leq 4 \]
\[ 0 \leq \theta \leq \pi. \]

Thus
\[ \mathbf{r}_x = \mathbf{i} \]
\[ \mathbf{r}_\theta = -2 \sin(\theta)j + 2 \cos(\theta)k \]
\[ \mathbf{r}_x \times \mathbf{r}_\theta = -2 \sin(\theta)k - 2 \cos(\theta)j. \]

This normal vector is orthogonal to \( S \) but points in the downward rather than upward direction, so we take the opposite vector,
\[ \mathbf{r}_\theta \times \mathbf{r}_x = 2 \cos(\theta)j + 2 \sin(\theta)k, \]
to be our normal vector and thus obtain
\[ \iiint_{S} F \cdot n \, d\sigma = \iiint_{D} x\mathbf{k} \cdot (2 \cos(\theta) \mathbf{j} + 2 \sin(\theta) \mathbf{k}) \, dA \]

\[ = \iiint_{D} 2x \sin(\theta) \, dA \]

\[ = \int_{0}^{\pi} \int_{1}^{4} 2x \sin(\theta) \, d\theta \, dx \]

\[ = \left( \int_{1}^{4} 2x \, dx \right) \left( \int_{0}^{\pi} \sin(\theta) \, d\theta \right) \]

\[ = (15)(2) = 30. \]

**Solution 2:** The surface \( S \) is a level surface of the function \( f(x, y, z) = y^2 + z^2 \) with \( f_z = 2z \neq 0 \) on this surface (except on part of the boundary, which is ok). We note that

\[ \nabla f = 2y \mathbf{j} + 2z \mathbf{k} \]

\[ |\nabla f \cdot \mathbf{k}| = |2z| = 2z \]

\[ \frac{\nabla f}{|\nabla f \cdot \mathbf{k}|} = \frac{y}{2} \mathbf{j} + \mathbf{k}. \]

Thus

\[ \iiint_{S} F \cdot n \, d\sigma = \iiint_{D} x\mathbf{k} \cdot \frac{\nabla f}{|\nabla f \cdot \mathbf{k}|} \, dA \]

\[ = \iiint_{D} x\mathbf{k} \cdot \left( \frac{y}{2} \mathbf{j} + \mathbf{k} \right) \, dA \]

\[ = \int_{1}^{4} \int_{-2}^{2} x \, dx \, dy \]

\[ = 30. \]