1. Let $C$ be the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$. Let $F$ be the vector field

$$F(x,y) = -\cos(x)i + y\cos(x)j.$$ 

The vector field and curve are pictured together below.

Compute the counterclockwise circulation of $F$ around $C$. It is up to you whether you would rather do this directly or use Green’s Theorem.

**Solution 1:** The circulation is

$$\int_C (Mdx + Ndy).$$

The functions $M$ and $N$ are

$$M = -\cos(x)$$

$$N = y\cos(x).$$

If we wish to compute the circulation directly, we need to parameterize the four line segments that make up $C$. The segment, $C_1$, on the bottom can be parameterized by
\[
x = x \quad y = 0 \\
dx = dx \quad dy = 0
\]
and we obtain
\[
\int_{C_1} (M \, dx + N \, dy) = \int_0^{\pi/2} -\cos(x) \, dx = -1.
\]
The segment, \(C_2\), on the right can be parameterized by
\[
x = \pi/2 \quad y = y \\
dx = 0 \quad dy = dy
\]
and we obtain
\[
\int_{C_2} (M \, dx + N \, dy) = \int_0^{\pi/2} y \cos\left(\frac{\pi}{2}\right) \, dy = 0.
\]
The segment, \(C_3\), on the top can be parameterized by
\[
x = x \quad y = \pi/2 \\
dx = dx \quad dy = 0
\]
and we obtain
\[
\int_{C_3} (M \, dx + N \, dy) = \int_{\pi/2}^{0} -\cos(x) \, dx = 1.
\]
The final segment, \(C_4\), can be parameterized by
\[
x = 0 \quad y = y \\
dx = 0 \quad dy = dy
\]
and we obtain
\[
\int_{C_4} (M \, dx + N \, dy) = \int_{0}^{\pi/2} N \, dy = \int_{\pi/2}^{0} y \cos(0) \, dy = -\frac{\pi^2}{8}.
\]
The circulation around \(C\) is thus
\[
-1 + 0 + 1 - \frac{\pi^2}{8} = -\frac{\pi^2}{8}.
\]
**Solution 2:** By Green’s Theorem, the circulation is
\[
\int_C (M \, dx + N \, dy) = \iint_D (N_x - M_y) \, dA
\]
\[
= \int_0^{\pi/2} \int_0^{\pi/2} (-y \sin(x) - 0) \, dy \, dx
\]
\[
= \left( \int_0^{\pi/2} y \, dy \right) \left( \int_0^{\pi/2} -\sin(x) \, dx \right)
\]
\[
= \frac{1}{2} \left( \frac{\pi}{2} \right)^2 (0 - 1)
\]
\[
= -\frac{\pi^2}{8}.
\]
Let $C$ be the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$. Let $F$ be the vector field
\[ F(x, y) = -\cos(x)i + y\cos(x)j. \]
The vector field and curve are pictured together in problem 1.
Compute the **outward flux** of $F$ across $C$. It is up to you whether you would rather do this directly or use Green’s Theorem.

**Solution 1**: The outward flux is
\[ \int_C (M\,dy - N\,dx). \]
The functions $M$ and $N$ are
\[ M = -\cos(x) \]
\[ N = y\cos(x). \]
If we wish to compute the flux directly, we need to parameterize the four line segments that make up $C$. The segment, $C_1$, on the bottom can be parameterized by
\[ x = x \quad y = 0 \]
\[ dx = dx \quad dy = 0 \]
and we obtain
\[ \int_{C_1} (M\,dy - N\,dx) = \int_0^{\pi/2} -\cos(x) \, dx = 0. \]
The segment, $C_2$, on the right can be parameterized by
\[ x = \pi/2 \quad y = y \]
\[ dx = 0 \quad dy = dy \]
and we obtain
\[ \int_{C_2} (M\,dy - N\,dx) = \int_0^{\pi/2} -\cos\left(\frac{\pi}{2}\right) \, dy = 0. \]
The segment, $C_3$, on the top can be parameterized by
\[ x = x \quad y = \pi/2 \]
\[ dx = dx \quad dy = 0 \]
and we obtain
\[ \int_{C_3} (M\,dy - N\,dx) = \int_{\pi/2}^0 -\frac{\pi}{2}\cos(x) \, dx = \frac{\pi}{2}. \]
The final segment, $C_4$, can be parameterized by
\[ x = 0 \quad y = y \]
\[ dx = 0 \quad dy = dy \]
and we obtain
\[
\int_{C_4} (M\,dy - N\,dx) = \int_{C_4} M\,dy = \int_{\pi/2}^{0} -\cos(0)\,dy = \frac{\pi}{2}.
\]

The flux across \(C\) is thus

\[0 + 0 + \frac{\pi}{2} + \frac{\pi}{2} = \pi.\]

**Solution 2:** By Green’s Theorem, the flux is

\[
\int_C (M\,dy - N\,dx) = \iint_D (M_x + N_y)\,dA
\]

\[
= \int_0^{\pi/2} \int_0^{\pi/2} (\sin(x) + \cos(x))\,dx\,dy
\]

\[
= \left(\int_0^{\pi/2} 1\,dy\right)\left(\int_0^{\pi/2} (\sin(x) + \cos(x))\,dx\right)
\]

\[
= \frac{\pi}{2} (-\cos(x) + \sin(x))\Big|_{x=0}^{x=\pi/2}
\]

\[
= \frac{\pi}{2} ((-0 + 1) - (-1 + 0))
\]

\[
= \pi.
\]

3. The cap of the sphere \(x^2 + y^2 + z^2 = 4\) that lies above the cone \(z = \sqrt{x^2 + y^2}\) is pictured below. The cone is also pictured.

Write parametric equations for the portion of the sphere pictured. (Note that it is necessary to include the ranges of parameter values that describe the given portion of the sphere.)

**Solution:** We know that the sphere (which has radius 2) can be parameterized as
\[ x = 2 \cos(\theta) \sin(\phi) \]
\[ y = 2 \sin(\theta) \sin(\phi) \]
\[ z = 2 \cos(\phi). \]

Since we want the whole cap, it is easy to see that we must take
\[ 0 \leq \theta \leq 2\pi. \]

On the sphere we have \( x^2 + y^2 + z^2 = 4 \) and on the cone we have \( x^2 + y^2 = z^2. \)
Therefore the sphere intersects the cone in the first octant when \( z^2 + z^2 = 4 \) or \( z = \sqrt{2}. \) This \( z \) value on the sphere corresponds to
\[ \cos(\phi) = \frac{\sqrt{2}}{2} \]
or \( \phi = \pi/4. \) We thus see that the cap of the sphere pictured corresponds to
\[ 0 \leq \phi \leq \frac{\pi}{4}. \]

4. Compute the surface area of the cap of the sphere given in the previous problem. You may use the fact that for a sphere of radius \( a, \) parameterized using spherical coordinates, it is known that \(|r_\theta \times r_\phi| = a^2 \sin(\phi)|. \)

**Solution:** Since the sphere has radius \( a = 2, \) the surface area is
\[ \int \int_D 4 \sin(\phi) \, dA \]

where
\[ D = \{(\theta, \phi) \mid 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \pi/4\}. \]

(This is known by the work done in the previous problem.) The surface area is thus
\[ \int_0^{2\pi} \int_0^{\pi/4} 4 \sin(\phi) \, d\phi \, d\theta = 4 \int_0^{2\pi} \left[-\cos(\phi)\right]_{\phi=0}^{\phi=\pi/4} \, d\theta \]
\[ = 4 \int_0^{2\pi} \left(-\frac{\sqrt{2}}{2} - (-1)\right) \, d\theta \]
\[ = 2 \left(2 - \sqrt{2}\right) \int_0^{2\pi} 1 \, d\theta \]
\[ = 4 \left(2 - \sqrt{2}\right) \pi. \]

5. Let \( S \) be the cylindrical surface (pictured)
\[ y^2 + z^2 = 4 \]
\[ z \geq 0 \]
\[ 1 \leq x \leq 4. \]
Evaluate \[ \iiint_S x \, d\sigma. \]

**Solution:** The surface \( S \) is

\[
\mathbf{r}(x, \theta) = xi + 2 \cos(\theta)j + 2 \sin(\theta)k
\]

\[
1 \leq x \leq 4
\]

\[
0 \leq \theta \leq \pi.
\]

Thus

\[
\mathbf{r}_x = i
\]

\[
\mathbf{r}_\theta = -2 \sin(\theta)j + 2 \cos(\theta)k
\]

\[
\mathbf{r}_x \times \mathbf{r}_\theta = -2 \sin(\theta)k - 2 \cos(\theta)j
\]

\[
|\mathbf{r}_x \times \mathbf{r}_\theta| = 2.
\]

We thus obtain
\[ \int_S x \, d\sigma = \int_D x |\mathbf{r}_x \times \mathbf{r}_\theta| \, dA \]
\[ = \int_D 2x \, dA \]
\[ = 2 \int_1^4 \int_0^\pi x \, d\theta \, dx \]
\[ = 2 \left( \int_1^4 x \, dx \right) \left( \int_0^\pi 1 \, d\theta \right) \]
\[ = 2 \left( \frac{15}{2} \right) \pi \]
\[ = 15\pi. \]

6. Let \( S \) be the cylindrical surface

\[ y^2 + z^2 = 4 \]
\[ z \geq 0 \]
\[ 1 \leq x \leq 4 \]

and let \( \mathbf{F} \) be the vector field

\[ \mathbf{F}(x,y,z) = z^2 \mathbf{k}. \]

The vector field and surface are pictured together below.

Find the flux of \( \mathbf{F} \) in the upward direction across \( S \).

**Solution 1:** The surface \( S \) is
\[ \mathbf{r}(x, \theta) = x\mathbf{i} + 2\cos(\theta)\mathbf{j} + 2\sin(\theta)\mathbf{k} \]
\[ 1 \leq x \leq 4 \]
\[ 0 \leq \theta \leq \pi. \]

Thus
\[ \mathbf{r}_x = \mathbf{i} \]
\[ \mathbf{r}_\theta = -2\sin(\theta)\mathbf{j} + 2\cos(\theta)\mathbf{k} \]
\[ \mathbf{r}_x \times \mathbf{r}_\theta = -2\sin(\theta)\mathbf{k} - 2\cos(\theta)\mathbf{j}. \]

This normal vector is orthogonal to \( S \) but points in the downward rather than upward direction, so we take the opposite vector,
\[ \mathbf{r}_\theta \times \mathbf{r}_x = 2\cos(\theta)\mathbf{j} + 2\sin(\theta)\mathbf{k}, \]
to be our normal vector and thus obtain
\[ \iint_{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_{D} 4\sin^2(\theta)\mathbf{k} \cdot (2\cos(\theta)\mathbf{j} + 2\sin(\theta)\mathbf{k}) \, dA \]
\[ = \iint_{D} 8\sin^3(\theta) \, dA \]
\[ = 8 \int_{1}^{4} \int_{0}^{\pi} (1 - \cos^2(\theta)) \sin(\theta) \, d\theta \, dx \]
\[ = 8 \left[ \int_{1}^{4} \left[ -\cos(\theta) + \frac{1}{3} \cos^3(\theta) \right] \right]_{\theta=0}^{\theta=\pi} \]
\[ = 8 \int_{1}^{4} \left( \left( 1 - \frac{1}{3} \right) - \left( -1 + \frac{1}{3} \right) \right) \, dx \]
\[ = 32. \]

**Solution 2**: The surface \( S \) is a level surface of the function \( f(x,y,z) = y^2 + z^2 \) with \( f_z = 2z \neq 0 \) on this surface (except on part of the boundary, which is ok). We note that
\[ \nabla f = 2y\mathbf{j} + 2z\mathbf{k} \]
\[ |\nabla f \cdot \mathbf{k}| = |2z| = 2z \]
\[ \frac{\nabla f}{|\nabla f \cdot \mathbf{k}|} = \frac{y}{z} \mathbf{j} + \mathbf{k}. \]

Thus
\[ \iiint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D z^2 \mathbf{k} \cdot \frac{\nabla f}{|\nabla f \cdot \mathbf{k}|} \, dA \]
\[ = \iiint_D z^2 \mathbf{k} \cdot \left( \frac{y}{z} \mathbf{j} + \mathbf{k} \right) \, dA \]
\[ = \int_1^4 \int_{-2}^2 z^2 \, dy \, dx \]
\[ = \int_1^4 \int_{-2}^2 (4 - y^2) \, dy \, dx \]
\[ = 32. \]