MATH 2203 – Exam 5 (Version 1) Solutions
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Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Match the graphs A–E with their contour (level curve) plots 1–5.
   Answers:
   Graph A matches contour plot ____2____.
   Graph B matches contour plot ____5____.
   Graph C matches contour plot ____4____.
   Graph D matches contour plot ____3____.
   Graph E matches contour plot ____1____.

2. Show that
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + 3y^2}
   \]
   does not exist.
   Be sure to be detailed and write in sentences.
   **Explanation:** We will show that the limit has different values for \((x, y) \to (0, 0)\) along different paths leading to \((0, 0)\). If we let \((x, y) \to (0, 0)\) along the path \(y = x\), then we obtain
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + 3y^2} = \lim_{x \to 0} \frac{x^2}{x^2 + 3x^2} = \lim_{x \to 0} \frac{x^2}{4x^2} = \frac{1}{4}.
   \]
   If we let \((x, y) \to (0, 0)\) along the path \(y = 2x\), then we obtain
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + 3y^2} = \lim_{x \to 0} \frac{x(2x)}{x^2 + 3(2x)^2} = \lim_{x \to 0} \frac{2x^2}{13x^2} = \frac{2}{13}.
   \]
   Therefore the limit in question does not exist.

3. A graph of the function
   \[
   f(x,y) = (4x - x^2)(4y - y^2),
   \]
   is shown below. The point \((3, 1, 9)\) is labelled on the graph.
a. Compute the partial derivative $f_x$ and then compute $f_x(3, 1)$.

b. Compute the partial derivative $f_y$ and then compute $f_y(3, 1)$.

c. On the graph that is provided, draw small tangent lines at the point $(3, 1, 9)$ to illustrate the results you found in parts a and b.

d. Do the results that you obtained in part a, b, and c makes sense to you in regard to the graph of $f$ at the point $(3, 1, 9)$? Explain why or why not.

**Answers:** We have

$$f_x = (4 - 2x)(4y - y^2)$$
$$f_y = (4x - x^2)(4 - 2y).$$

Also

$$f_x(3, 1) = (4 - 2(3))(4(1) - 1^2) = -6$$
$$f_y(3, 1) = (4(3) - 3^2)(4 - 2(1)) = 6.$$

The value of $f_x(3, 1)$ is the slope of the tangent line to the graph of $z = f(x, 1)$ at the point $(3, 1, 9)$. The value of $f_y(3, 1)$ is the slope of the tangent line to the graph of $z = f(3, y)$ at the point $(3, 1, 9)$. These tangent lines are illustrated below and they do make sense because we can see in the picture that $f$ has negative slope in the $x$ direction at the point $(3, 1, 9)$ and has positive slope in the $y$ direction at $(3, 1, 9)$. 

4. For the function \( f \) given in Question 3:
   a. Compute \( f_{xx} \) and \( f_{yy} \) and then compute \( f_{xx}(3, 1) \) and \( f_{yy}(3, 1) \)
   b. Explain what the values of \( f_{xx}(3, 1) \) and \( f_{yy}(3, 1) \) tell you about the graph of \( f \) at the point \( (3, 1, 9) \).
   c. Compute \( f_{xy} \) and \( f_{yx} \).
   d. You should have found in part c that \( f_{xy} = f_{yx} \). Is this unusual? Explain.

**Answers:** In Question 3 we found

\[
\begin{align*}
f_x &= (4 - 2x)(4y - y^2) \\
f_y &= (4x - x^2)(4 - 2y).
\end{align*}
\]

From this we obtain

\[
\begin{align*}
f_{xx} &= -2(4y - y^2) \\
f_{yy} &= -2(4x - x^2)
\end{align*}
\]

and

\[
\begin{align*}
f_{xx}(3, 1) &= -2(4(1) - 1^2) = -6 \\
f_{yy}(3, 1) &= -2(4(3) - 3^2) = -6.
\end{align*}
\]

These values, since they are both negative, tell us that the graph of \( f \) is concave down in both the \( x \) direction and the \( y \) direction at the point \( (3, 1, 9) \). This can be seen to be correct by looking at the picture given in Question 3.

Also

\[
\begin{align*}
f_{xy} &= (4 - 2x)(4 - 2y) \\
f_{yx} &= (4 - 2x)(4 - 2y).
\end{align*}
\]

It turns out that \( f_{xy} = f_{yx} \) as expected because \( f_{xy} \) and \( f_{yx} \) are both continuous functions and thus Clairaut’s Theorem guarantees that \( f_{xy} = f_{yx} \).

5. Suppose that
\[ z = e^{x^2+3y} \]
\[ x = \cos(s) - \sin(t) \]
\[ y = 3e^{s+t}. \]

Compute \( \partial z/\partial s \) and then find the value of \( \frac{\partial z}{\partial s} \bigg|_{(s,t)=(0,0)} \)

**Solution:** By the Chain Rule,
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
\[
= \left(2xe^{x^2+3y}\right)(-\sin(s)) + \left(3e^{x^2+3y}\right)(3e^{s+t})
\]
\[
= -\left(2xe^{x^2+3y}\right)(\sin(s)) + \left(3e^{x^2+3y}\right)(3e^{s+t}).
\]

Notice that when \( (s,t) = (0,0) \) we have
\[
\begin{align*}
x &= \cos(0) - \sin(0) = 1 \\
y &= 3e^{0+0} = 3
\end{align*}
\]
and thus
\[
\frac{\partial z}{\partial s} \bigg|_{(s,t)=(0,0)} = -\left(2(1)e^{12+3(3)}\right)(\sin(0)) + \left(3e^{12+3(3)}\right)(3e^{0+0}) = 9e^{10}.
\]

6. For the function
\[ f(x,y) = (4x-x^2)(4y-y^2) \]
(whose graph is shown in Question 3):

a. Find \( \nabla f(x,y) \).
b. Find \( \nabla f(3,1) \).
c. In what direction is \( f \) increasing most rapidly at \( (x,y) = (3,1) \)? (Give the unit vector, \( u \), that points in the direction of greatest increase.)
d. For the unit vector, \( u \), that you gave in part c, what is the value of \( D_u f(3,1) \)?

**Answers:** We have
\[
\nabla f(x,y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = (4-2x)(4y-y^2) \mathbf{i} + (4x-x^2)(4-2y) \mathbf{j}
\]
and
\[
\nabla f(3,1) = \frac{\partial f}{\partial x}(3,1) \mathbf{i} + \frac{\partial f}{\partial y}(3,1) \mathbf{j} = -6 \mathbf{i} + 6 \mathbf{j}.
\]

We know that \( f \) increases most rapidly at \( (x,y) = (3,1) \) in the direction of \( \nabla f(3,1) \). The unit vector pointing in this direction is
\[
\mathbf{u} = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j}).
\]

In this direction we have
\[
D_u f(3,1) = |\nabla f(3,1)| = |-6 \mathbf{i} + 6 \mathbf{j}| = 6|\mathbf{i} + \mathbf{j}| = 6\sqrt{2}.
\]