1. Match the graphs A–E with their contour (level curve) plots 1–5.
   Answers:
   Graph A matches contour plot ___4____.
   Graph B matches contour plot ___3____.
   Graph C matches contour plot ___2____.
   Graph D matches contour plot ___1____.
   Graph E matches contour plot ___5____.

2. Show that
   \[ \lim_{(x,y) \to (0,0)} \frac{x^4}{x^2y^2 + 3y^4} \]
   does not exist.

   Be sure to be detailed and write in sentences.

   **Explanation:** We will show that the limit has different values for \((x,y) \to (0,0)\) along different paths leading to \((0,0)\). If we let \((x,y) \to (0,0)\) along the path \(y = x\), then we obtain
   \[
   \lim_{y \to x, (x,y) \to (0,0)} \frac{x^4}{x^2y^2 + 3y^4} = \lim_{x \to 0} \frac{x^4}{x^2x^2 + 3x^4} = \lim_{x \to 0} \frac{x^4}{4x^4} = \frac{1}{4}.
   \]

   If we let \((x,y) \to (0,0)\) along the path \(x = 0\), then we obtain
   \[
   \lim_{y \to 0, (x,y) \to (0,0)} \frac{x^4}{x^2y^2 + 3y^4} = \lim_{y \to 0} \frac{0^4}{0^2y^2 + 3y^4} = \lim_{y \to 0} 0 = 0.
   \]

   Therefore the limit in question does not exist.

3. A graph of the function
   \[ f(x,y) = (4x - x^2)(4y - y^2), \]
   is shown below. The point (2, 1, 12) is labelled on the graph.
a. Compute the partial derivative $f_x$ and then compute $f_x(2, 1)$.

b. Compute the partial derivative $f_y$ and then compute $f_y(2, 1)$.

c. On the graph that is provided, draw small tangent lines at the point $(2, 1, 12)$ to illustrate the results you found in parts a and b.

d. Do the results that you obtained in part a, b, and c makes sense to you in regard to the graph of $f$ at the point $(2, 1, 12)$? Explain why or why not.

**Answers:**

We have

\[
\begin{align*}
  f_x &= (4 - 2x)(4y - y^2) \\
  f_y &= (4x - x^2)(4 - 2y).
\end{align*}
\]

Also

\[
\begin{align*}
  f_x(2, 1) &= (4 - 2(2))(4(1) - 1^2) = 0 \\
  f_y(2, 1) &= (4(2) - 2^2)(4 - 2(1)) = 8.
\end{align*}
\]

The value of $f_x(2, 1)$ is the slope of the tangent line to the graph of $z = f(x, 1)$ at the point $(2, 1, 12)$. The value of $f_y(2, 1)$ is the slope of the tangent line to the graph of $z = f(2, y)$ at the point $(2, 1, 12)$. These tangent lines are illustrated below and they do make sense because we can see in the picture that $f$ has zero slope in the $x$ direction at the point $(2, 1, 12)$ and has positive slope in the $y$ direction at $(2, 1, 12)$.
4. For the function $f$ given in Question 3:
   a. Compute $f_{xx}$ and $f_{yy}$ and then compute $f_{xx}(2, 1)$ and $f_{yy}(2, 1)$
   b. Explain what the values of $f_{xx}(2, 1)$ and $f_{yy}(2, 1)$ tell you about the graph of $f$ at the point $(2, 1, 12)$.
   c. Compute $f_{xy}$ and $f_{yx}$.
   d. You should have found in part c that $f_{xy} = f_{yx}$. Is this unusual? Explain.

**Answers:** In Question 3 we found

$$f_x = (4 - 2x)(4y - y^2)$$
$$f_y = (4x - x^2)(4 - 2y).$$

From this we obtain

$$f_{xx} = -2(4y - y^2)$$
$$f_{yy} = -2(4x - x^2)$$

and

$$f_{xx}(2, 1) = -2(4(1) - 1^2) = -6$$
$$f_{yy}(2, 1) = -2(4(2) - 2^2) = -8$$

These values, since they are both negative, tell us that the graph of $f$ is concave down in both the $x$ direction and the $y$ direction at the point $(2, 1, 12)$. This can be seen to be correct by looking at the picture given in Question 3. Also

$$f_{xy} = (4 - 2x)(4 - 2y)$$
$$f_{yx} = (4 - 2x)(4 - 2y).$$

It turns out that $f_{xy} = f_{yx}$ as expected because $f_{xy}$ and $f_{yx}$ are both continuous functions and thus Clairaut’s Theorem guarantees that $f_{xy} = f_{yx}$.

5. Suppose that
\[ z = e^{x^2+y} \]
\[ x = \cos(s) - \sin(t) \]
\[ y = 3e^{s+t}. \]

Compute \( \frac{\partial z}{\partial t} \) and then find the value of \( \frac{\partial z}{\partial t} \bigg|_{(s,t) = (0,0)} \).

**Solution:** By the Chain Rule,
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \\
= \left(2xe^{x^2+y}\right)(-\cos(t)) + \left(3e^{x^2+y}\right)(3e^{s+t}) \\
= -\left(2xe^{x^2+y}\right)(\cos(t)) + \left(3e^{x^2+y}\right)(3e^{s+t}).
\]
Notice that when \((s,t) = (0,0)\) we have
\[
x = \cos(0) - \sin(0) = 1 \\
y = 3e^{0+0} = 3
\]
and thus
\[
\frac{\partial z}{\partial t} \bigg|_{(s,t) = (0,0)} = -\left(2(1)e^{1^2+3(0)}\right)(\cos(0)) + \left(3e^{1^2+3(0)}\right)(3e^{0+0}) = 7e^{10}.
\]

6. For the function
\[
f(x,y) = (4x - x^2)(4y - y^2)
\]
(whose graph is shown in Question 3):
   a. Find \( \nabla f(x,y) \).
   b. Find \( \nabla f(2,1) \).
   c. In what direction is \( f \) increasing most rapidly at \((x,y) = (2,1)\)? (Give the unit vector, \( \mathbf{u} \), that points in the direction of greatest increase.)
   d. For the unit vector, \( \mathbf{u} \), that you gave in part c, what is the value of \( D_u f(2,1) \)?

**Answers:** We have
\[
\nabla f(x,y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = (4 - 2x)(4y - y^2) \mathbf{i} + (4x - x^2)(4 - 2y) \mathbf{j}
\]
and
\[
\nabla f(2,1) = \frac{\partial f}{\partial x}(2,1) \mathbf{i} + \frac{\partial f}{\partial y}(2,1) \mathbf{j} = 8\mathbf{j}.
\]
We know that \( f \) increases most rapidly at \((x,y) = (2,1)\) in the direction of \( \nabla f(2,1) \). The unit vector pointing in this direction is \( \mathbf{u} = \mathbf{j} \). In this direction we have
\[
D_u f(2,1) = |\nabla f(2,1)| = |8\mathbf{j}| = 8|\mathbf{j}| = 8.
\]