Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

Note: I have not included diagrams with any of these solutions. Please let me know if you are having trouble with drawing the diagrams.

1. (An all–or–none question: You either get 20 points or 0 points on this one.) In \( \mathbb{R}^3 \), the set of points described by the three equations (taken together)
   \[
   \begin{align*}
   x &= -3 \\
   y &= -3 \\
   z &= 0
   \end{align*}
   \]
   is (circle the correct choice):
   
   a. a single point.
   
   b. a single line.
   
   c. a single plane.
   
   d. three points.
   
   e. three lines.
   
   f. three planes.
   
   g. None of the above.

2. An airplane flying in the direction 30° north of west at 500 miles per hour encounters a 100 mile per hour wind blowing in the direction 60° north of east. The airplane holds it compass heading 30° north of west but, because of the wind, acquires a new ground speed and direction. What are they? (Show you work in detail. Include diagrams, all relevant calculations, and narrative explaining your reasoning.)

Solution: The air velocity vector of the plane is
   \[ V = 500 \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle -250\sqrt{3}, 250 \rangle. \]

The wind velocity vector is
   \[ W = 100 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle 50, 50\sqrt{3} \rangle. \]

The newly-acquired ground velocity vector is
   \[ G = V + W = \langle -250\sqrt{3} + 50, 250 + 50\sqrt{3} \rangle. \]

The ground speed of the plane is thus
\[ |\mathbf{G}| = \sqrt{(-250\sqrt{3} + 50)^2 + (250 + 50\sqrt{3})^2} \approx 509.9 \text{ miles per hour} \]

and the direction is
\[ 30^\circ + \arctan\left(\frac{100}{500}\right) \approx 41.31^\circ \]
north of west.

3. Find the vector, \( \mathbf{v} \), that has magnitude 2 and that points in the same direction as the vector \( 4\mathbf{i} + 3\mathbf{j} \).

**Solution:** Since
\[ |4\mathbf{i} + 3\mathbf{j}| = \sqrt{4^2 + 3^2} = 5, \]
the unit vector that points in the same direction as \( 4\mathbf{i} + 3\mathbf{j} \) is
\[ \frac{1}{5}(4\mathbf{i} + 3\mathbf{j}) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}. \]
Therefore the vector of magnitude 2 that points in the same direction as \( 4\mathbf{i} + 3\mathbf{j} \) is
\[ \mathbf{v} = 2\left(\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}\right) = \frac{8}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}. \]

4. For the vectors
\[ \mathbf{u} = \langle 4, 2 \rangle \quad \text{and} \quad \mathbf{v} = \langle -3, 3 \rangle, \]

a. Draw \( \mathbf{u} \) and \( \mathbf{v} \) in standard position.

b. Compute \( \mathbf{u} \cdot \mathbf{v} \).

c. Compute \( |\mathbf{u}| \) and \( |\mathbf{u}| \).

d. Find the angle, \( \theta \), between \( \mathbf{u} \) and \( \mathbf{v} \).

e. Find \( \text{proj}_v \mathbf{u} \) and include a picture of it with the picture you drew in part a.

**Solution:** We have
\[ \mathbf{u} \cdot \mathbf{v} = (4)(-3) + (2)(3) = -6 \]
\[ |\mathbf{u}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5} \]
\[ |\mathbf{v}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}. \]

Thus
\[ \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-6}{(2\sqrt{5})(3\sqrt{2})} = \frac{-1}{(\sqrt{5})(\sqrt{2})} = -\frac{\sqrt{10}}{10} \]
and we see that
\[ \theta = \arccos\left(-\frac{\sqrt{10}}{10}\right) \approx 108.43^\circ. \]

Next we note that
\[ \text{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{-6}{18} \langle -3, 3 \rangle = -\frac{1}{3} \langle -3, 3 \rangle = \langle 1, -1 \rangle. \]

5. For the vectors
\[ \mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \quad \text{and} \quad \mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \]
a. Compute $u \times v$.

b. Show that $u \times v$ is orthogonal to both $u$ and $v$.

Solution:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 2 & -4 \\ -4 & 3 & 2 \end{vmatrix}$$

$$= ((2)(2) - (-4)(3))i - ((2)(2) - (-4)(-4))j + ((2)(3) - (2)(-4))k$$

$$= 16i + 12j + 14k.$$ 

To see that $u \times v$ is orthogonal to both $u$ and $v$, we note that

$$(u \times v) \cdot u = (16i + 12j + 14k) \cdot (2i + 2j - 4k)$$

$$= (16)(2) + (12)(2) + (14)(-4)$$

$$= 0$$

and

$$(u \times v) \cdot v = (16i + 12j + 14k) \cdot (-4i + 3j + 2k)$$

$$= (16)(-4) + (12)(3) + (14)(2)$$

$$= 0.$$ 

6. Find both parametric and symmetric equations of the line, $L$, that contains the points $P_0(-2, 2, -1)$ and $P_1(4, 1, 4)$.

Solution: A direction vector for the line, $L$, is

$$v = \langle 4 - (-2), 1 - 2, 4 - (-1) \rangle = \langle 6, -1, 5 \rangle.$$ 

Thus parametric equations for $L$ are

$$x = -2 + 6t$$

$$y = 2 - t$$

$$z = -1 + 5t$$

$$-\infty < t < \infty.$$ 

When we solve the first to equations above for $t$, we obtain

$$t = \frac{x + 2}{6}$$

$$t = 2 - y$$

$$t = \frac{z + 1}{5}.$$ 

Thus, symmetric equations for $L$ are

$$\frac{x + 2}{6} = 2 - y = \frac{z + 1}{5}$$

which can also be written as

$$5x + 10 = 60 - 30y = 6z + 6.$$