1. The position vector, at time $t$, of a particle in motion in the $xy$ plane is given by
   \[ r(t) = t\mathbf{i} - t^2\mathbf{j}. \]
   
   a. Find the path of motion of the particle. (Explain and draw a picture.)
   
   b. Find the velocity vector, $v(t)$, at any time $t$.
   
   c. Find the speed at any time $t$.
   
   d. Find the acceleration vector, $a(t)$, at any time $t$.
   
   e. In the picture that you drew in part a, draw the position vector and the velocity vector at time $t = 0$. Do the same at time $t = 2$.

   **Solution:** Note that when we write the position function in parametric form, we obtain
   \[ x = t \]
   \[ y = -t^2. \]
   Since, at all times $t$, we have $y = -t^2 = -x^2$, we see that the path of motion of the particle is the parabola $y = -x^2$.

   The velocity vector is
   \[ v(t) = \mathbf{i} - 2t\mathbf{j} \]
   and the speed is
   \[ v(t) = |v(t)| = \sqrt{1^2 + (-2t)^2} = \sqrt{1 + 4t^2}. \]
   The acceleration vector is
   \[ a(t) = -2\mathbf{j}. \]

   The position and velocity vectors at time $t = 0$ and $t = 2$ are as follows:
   \[ r(0) = 0\mathbf{i} - 0^2\mathbf{j} = 0 \]
   \[ v(0) = \mathbf{i} - 2(0)\mathbf{j} = \mathbf{i} \]
   \[ r(2) = 2\mathbf{i} - 2^2\mathbf{j} = 2\mathbf{i} - 4\mathbf{j} \]
   \[ v(2) = \mathbf{i} - 2(2)\mathbf{j} = \mathbf{i} - 4\mathbf{j}. \]
   This is illustrated below.
2. The velocity vector, at time $t$, of a particle in motion in the $xy$ plane is given by
\[ \mathbf{v}(t) = \cos(t)\mathbf{i} + \cos(t)\mathbf{j} \]

and the particle is located at the point \((0, 2)\) at time \(t = 0\).

**a.** Find the position vector of the particle at time \(t\).

**b.** Where is the particle located at times \(t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{and } 2\pi\)? (Fill in the table below.)

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\pi)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{3\pi}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2\pi)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Sketch the path of motion of the particle.

**Solution:** The position vector is

\[ \mathbf{r}(t) = \sin(t)\mathbf{i} + \sin(t)\mathbf{j} + \mathbf{C} \]

where \(\mathbf{C}\) is a constant vector. At time \(t = 0\) we have

\[ \mathbf{r}(0) = \mathbf{C} = 2\mathbf{j}. \]

Thus

\[ \mathbf{r}(t) = \sin(t)\mathbf{i} + (\sin(t) + 2)\mathbf{j}. \]

Parametric equations for the path of motion of the particle are

\[ x = \sin(t) \]
\[ y = \sin(t) + 2. \]

The following table shows the location of the particle at the requested times.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\frac{\pi}{2})</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\frac{3\pi}{2})</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(2\pi)</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Since at all times \(t\) we have \(y = \sin(t) + 2 = x + 2\), we see that the path of motion is the line segment \(y = x + 2\) between the points \((-1, 1)\) and \((1, 3)\). This is pictured below.
3. A woman stands 20 feet away from a box of width 5 feet and height 3 feet (See Diagram) and tosses a bean bag – hoping to get it to land in the box. She tosses the bean bag from an initial height of 4 feet and with an initial angle of $\alpha = 0^\circ$. What range of initial tossing speeds ($v_0$) will ensure that the bean bag lands in the box? (Assume that the only vertical force that acts on the bean bags during their flight is the acceleration due to gravity. Thus assume that $a(t) = -32 \text{j}$.) Be detailed in your solution. Write in sentences.
Solution: Since $a(t) = -32j$, we have

$$v(t) = -32tj + C$$

where $C$ is a constant vector. Also, if the initial speed is $v_0$, then the initial velocity vector is

$$v_0 = v_0 \cos(0^\circ)i + v_0 \sin(0^\circ)j = v_0i.$$  

Since

$$v(0) = C = v_0i,$$

we have

$$v(t) = v_0i - 32tj.$$  

The position vector at time $t$ is

$$r(t) = v_0ti - 16t^2j + C$$

where $C$ is a constant vector.  

Since

$$r(0) = C = 4j,$$

we have

$$r(t) = v_0ti + (4 - 16t^2)j.$$  

Parametric equations for the path of motion are

$$x(t) = v_0t$$

and

$$y(t) = 4 - 16t^2.$$  

In order for the bean bag to get into the box, we must have a value of $t$ such that

$$20 \leq x(t) \leq 25$$

and

$$y(t) = 3.$$  

The equation $y(t) = 3$ is the same as

$$4 - 16t^2 = 3$$

or
and this equation holds when \( t = 1/4 \). At this value of \( t \) we have
\[
x(t) = \frac{1}{4} v_0.
\]
In order for the bean bag to land in the box, we must thus have
\[
20 \leq \frac{1}{4} v_0 \leq 25.
\]
We see that the bean bag will land in the box if the initial speed with which it is thrown is between 80 and 100 feet per second. The picture below shows the case where it is thrown with a speed of \( v_0 = 90 \) feet per second.

4. A particle is in motion according to
\[
x = t \\
y = e^t.
\]
(Note that the path of motion, which is pictured below, is the graph of \( y = e^x \).)

Find the tangential and normal components of acceleration (\( a_T(t) \) and \( a_N(t) \)) for this particle. *Hint:* Remember that you can do this without actually finding the curvature.
function for the curve. This is the easiest way.

**Solution:** First we will find the tangential component of acceleration. We have

\[ \mathbf{r}(t) = t \mathbf{i} + e^t \mathbf{j} \]
\[ \mathbf{v}(t) = \mathbf{i} + e^t \mathbf{j} \]
\[ v(t) = |\mathbf{v}(t)| = \sqrt{1^2 + (e^t)^2} = \sqrt{1 + e^{2t}}. \]

The tangential component of acceleration is thus

\[ a_T(t) = v'(t) = \frac{1}{2} (1 + e^{2t})^{-1/2} (2e^{2t}) = \frac{e^t}{\sqrt{1 + e^{2t}}}. \]

Now we will find the normal component of acceleration the **hard way** (which involves computing curvature). The unit tangent vector is

\[ \mathbf{T}(t) = \frac{1}{|\mathbf{v}(t)|} \mathbf{v}(t) = \frac{1}{\sqrt{1 + e^{2t}}} (\mathbf{i} + e^t \mathbf{j}) = (1 + e^{2t})^{-1/2} (\mathbf{i} + e^t \mathbf{j}) \]

and thus

\[ \mathbf{T}'(t) = (1 + e^{2t})^{-1/2} (e^t \mathbf{j}) - \frac{1}{2} (1 + e^{2t})^{-3/2} (2e^{2t})(\mathbf{i} + e^t \mathbf{j}) \]
\[ = (1 + e^{2t})^{-3/2} ((1 + e^2t)e^t \mathbf{j} - e^{2t}(\mathbf{i} + e^t \mathbf{j})) \]
\[ = (1 + e^{2t})^{-3/2} (-e^{2t} \mathbf{i} + e^t \mathbf{j}) \]
\[ = \frac{-e^t}{(1 + e^{2t})^{3/2}} (e^t \mathbf{i} - \mathbf{j}). \]

and

\[ |\mathbf{T}'(t)| = \left| \frac{-e^t}{(1 + e^{2t})^{3/2}} \right| |e^t \mathbf{i} - \mathbf{j}| = \left( \frac{e^t}{(1 + e^{2t})^{3/2}} \right) (1 + e^{2t})^{1/2} = \frac{e^t}{1 + e^{2t}}. \]

The curvature is thus

\[ \kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)} = \frac{e^t}{(1 + e^{2t})^{3/2}}. \]

We now see that the normal component of acceleration is

\[ a_N(t) = \kappa(t)v(t)^2 = \frac{e^t}{(1 + e^{2t})^{3/2}} (1 + e^{2t}) = \frac{e^t}{\sqrt{1 + e^{2t}}}. \]

To find the normal component of acceleration the **easy way**, we use the fact that

\[ a_N(t) = \sqrt{|\mathbf{a}(t)|^2 - a_T(t)^2}. \]

Since the acceleration vector is

\[ \mathbf{a}(t) = e^t \mathbf{j}, \]

we have

\[ |\mathbf{a}(t)| = e^t \]

and thus
\[ a_N(t) = \sqrt{|\mathbf{a}(t)|^2 - a_T(t)^2} \]

\[ = \sqrt{e^{2t} - \frac{e^{4t}}{1 + e^{2t}}} \]

\[ = \sqrt{(1 + e^{2t})e^{2t} - e^{4t}} \]

\[ = \sqrt{\frac{e^{2t}}{1 + e^{2t}}} \]

\[ = \frac{e^t}{\sqrt{1 + e^{2t}}}. \]