MATH 2203 - Exam 2 (Version 2) Solutions
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Instructions. Your work on this exam will be graded according to two criteria: **mathematical correctness** and **clarity of presentation**. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using **complete sentences** where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. The position vector, at time \( t \), of a particle in motion in the \( xy \) plane is given by
   \[
   \mathbf{r}(t) = -t\mathbf{i} - t^2\mathbf{j}.
   \]

   a. Find the path of motion of the particle. (Explain and draw a picture.)

   b. Find the velocity vector, \( \mathbf{v}(t) \), at any time \( t \).

   c. Find the speed at any time \( t \).

   d. Find the acceleration vector, \( \mathbf{a}(t) \), at any time \( t \).

   e. In the picture that you drew in part a, draw the position vector and the velocity vector at time \( t = 0 \). Do the same at time \( t = 2 \).

   **Solution:** Note that when we write the position function in parametric form, we obtain
   \[
   x = -t \\
y = -t^2.
   \]

   Since, at all times \( t \), we have \( y = -t^2 = -(t)^2 = -x^2 \), we see that the path of motion of the particle is the parabola \( y = -x^2 \).

   The velocity vector is
   \[
   \mathbf{v}(t) = -\mathbf{i} - 2t\mathbf{j}
   \]

   and the speed is
   \[
   v(t) = |\mathbf{v}(t)| = \sqrt{(-1)^2 + (-2t)^2} = \sqrt{1 + 4t^2}.
   \]

   The acceleration vector is
   \[
   \mathbf{a}(t) = -2\mathbf{j}.
   \]

   The position and velocity vectors at time \( t = 0 \) and \( t = 2 \) are as follows:
   \[
   \mathbf{r}(0) = -0\mathbf{i} - 0^2\mathbf{j} = 0 \\
   \mathbf{v}(0) = -\mathbf{i} - 2(0)\mathbf{j} = -\mathbf{i} \\
   \mathbf{r}(2) = -2\mathbf{i} - 2^2\mathbf{j} = -2\mathbf{i} - 4\mathbf{j} \\
   \mathbf{v}(2) = -\mathbf{i} - 2(2)\mathbf{j} = -\mathbf{i} - 4\mathbf{j}.
   \]

   This is illustrated below.
2. The velocity vector, at time $t$, of a particle in motion in the $xy$ plane is given by
\[ v(t) = \cos(t)i - \cos(t)j \]

and the particle is located at the point \((0, 2)\) at time \(t = 0\).

a. Find the position vector of the particle at time \(t\).

b. Where is the particle located at times \(t = \pi/2, \pi, 3\pi/2\) and \(2\pi\)? (Fill in the table below.)

c. Sketch the path of motion of the particle.

**Solution:** The position vector is

\[ r(t) = \sin(t)i - \sin(t)j + C \]

where \(C\) is a constant vector. At time \(t = 0\) we have

\[ r(0) = C = 2j. \]

Thus

\[ r(t) = \sin(t)i + (2 - \sin(t))j. \]

Parametric equations for the path of motion of the particle are

\[
\begin{align*}
  x &= \sin(t) \\
  y &= 2 - \sin(t).
\end{align*}
\]

The following table shows the location of the particle at the requested times.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\pi/2)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\pi)</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(3\pi/2)</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>(2\pi)</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Since at all times \(t\) we have \(y = 2 - \sin(t) = 2 - x\), we see that the path of motion is the line segment \(y = -x + 2\) between the points \((-1, 3)\) and \((1, 1)\). This is pictured below.
3. A woman who is 6 feet tall stands 20 feet away from a box of width 5 feet and height 3 feet (See Diagram) and tosses a bean bag – hoping to get it to land in the box. She tosses the bean bag from an initial height of 4 feet. Since she is not very good at tossing bean bags, she accidentally tosses the bean bag at an initial angle of $\alpha = 90^\circ$. The bean bag goes straight up and comes back down and hits her on the head 4 seconds after she first tosses it. (Remember that she is 6 feet tall, so the top of her head is at the point $(0,6)$.)

With what initial speed ($v_0$) did she toss the bean bag? (Assume that the only vertical force that acts on the bean bags during their flight is the acceleration due to gravity. Thus assume that $a(t) = -32 \hat{j}$.) Be detailed in your solution. Write in sentences.
Solution: Since \( a(t) = -32j \), we have
\[
\mathbf{v}(t) = -32t\mathbf{j} + \mathbf{C}
\]
where \( \mathbf{C} \) is a constant vector. Also, if the initial speed is \( v_0 \), then the initial velocity vector is
\[
\mathbf{v}_0 = v_0 \cos(90^\circ)\mathbf{i} + v_0 \sin(90^\circ)\mathbf{j} = v_0\mathbf{j}.
\]
Since
\[
\mathbf{v}(0) = \mathbf{C} = v_0\mathbf{j},
\]
we have
\[
\mathbf{v}(t) = (v_0 - 32t)\mathbf{j}.
\]
The position vector at time \( t \) is
\[
\mathbf{r}(t) = (v_0t - 16t^2)\mathbf{j} + \mathbf{C}
\]
where \( \mathbf{C} \) is a constant vector.
Since
\[
\mathbf{r}(0) = \mathbf{C} = 4\mathbf{j},
\]
we have
\[
\mathbf{r}(t) = (v_0t - 16t^2 + 4)\mathbf{j}.
\]
Parametric equations for the path of motion are
\[
x(t) = 0
\]
\[
y(t) = -16t^2 + v_0t + 4.
\]
In order for the bean bag to reach the point \((0, 6)\) at time \( t = 4 \) we must have \( y(4) = 6 \) which is the same as
\[
-16(4)^2 + 4v_0 + 4 = 6
\]
or
\[
v_0 = 64.5.
\]
To verify that this is correct, note that if \( y(t) = -16t^2 + 64.5t + 4 \), then \( y(0) = 4 \) and
4. A particle is in motion according to
\[ x = t \]
\[ y = \ln(t) \]
\[ t > 0. \]
(Note that the path of motion, which is pictured below, is the graph of \( y = \ln(x) \).)

Find the tangential and normal components of acceleration \((a_T(t) \text{ and } a_N(t))\) for this particle. **Hint:** Remember that you can do this without actually finding the curvature function for the curve. This is the easiest way.

**Solution:** First we will find the tangential component of acceleration. We have
\[ \mathbf{r}(t) = t \mathbf{i} + \ln(t) \mathbf{j} \]
\[ \mathbf{v}(t) = 1 \mathbf{i} + \frac{1}{t} \mathbf{j} \]
\[ v(t) = |\mathbf{v}(t)| = \sqrt{1^2 + \left(\frac{1}{t}\right)^2} = \frac{\sqrt{1 + t^2}}{t}. \]

The tangential component of acceleration is thus
\[ a_T(t) = v'(t) = \frac{t \cdot \frac{1}{2} (1 + t^2)^{-1/2} (2t) - (1 + t^2)^{1/2}}{t^2} \]
\[ = \frac{t^2 (1 + t^2)^{-1/2} - (1 + t^2)^{1/2}}{t^2} \]
\[ = (1 + t^2)^{-1/2} \cdot \left( \frac{t^2 - (1 + t^2)}{t^2} \right) \]
\[ = \frac{-1}{t^2 \sqrt{1 + t^2}} \]
Now we will find the normal component of acceleration the **hard way** (which involves computing curvature). The unit tangent vector is

\[
T(t) = \frac{1}{|v(t)|}v(t) = \frac{t}{\sqrt{1 + t^2}} \left( t\mathbf{i} + \frac{1}{t}\mathbf{j} \right)
\]

\[
= \frac{1}{\sqrt{1 + t^2}} (t\mathbf{i} + 1\mathbf{j})
\]

\[
= (1 + t^2)^{-1/2} (t\mathbf{i} + 1\mathbf{j})
\]

and thus

\[
T'(t) = (1 + t^2)^{-1/2} (i) - \frac{1}{2} (1 + t^2)^{-3/2} (2t)(t\mathbf{i} + 1\mathbf{j})
\]

\[
= (1 + t^2)^{-3/2} ((1 + t^2)(i) - t(t\mathbf{i} + 1\mathbf{j}))
\]

\[
= (1 + t^2)^{-3/2} (i - t\mathbf{j})
\]

and

\[
|T'(t)| = (1 + t^2)^{-3/2} |i - t\mathbf{j}| = \frac{1}{1 + t^2}.
\]

The curvature is thus

\[
\kappa(t) = \frac{|T'(t)|}{v(t)} = \frac{1}{1 + t^2} \left( \frac{t}{(1 + t^2)^{1/2}} \right) = \frac{t}{(1 + t^2)^{3/2}}.
\]

We now see that the normal component of acceleration is

\[
a_N(t) = \kappa(t)v(t)^2 = \left( \frac{t}{(1 + t^2)^{3/2}} \right) \left( \frac{1 + t^2}{t^2} \right) = \frac{1}{t\sqrt{1 + t^2}}.
\]

To find the normal component of acceleration the **easy way**, we use the fact that

\[
a_N(t) = \sqrt{|a(t)|^2 - a_T(t)^2}.
\]

Since the acceleration vector is

\[
a(t) = -\frac{1}{t^2}\mathbf{j},
\]

we have

\[
|a(t)| = \frac{1}{t^2}
\]

and thus
\[ a_N(t) = \sqrt{|a(t)|^2 - a_T(t)^2} \]
\[ = \sqrt{\frac{1}{t^4} - \frac{1}{t^4(1 + t^2)}} \]
\[ = \sqrt{\frac{(1 + t^2) - 1}{t^4(1 + t^2)}} \]
\[ = \sqrt{\frac{1}{t^2(1 + t^2)}} \]
\[ = \frac{1}{t\sqrt{1 + t^2}}. \]