Parametric Equations of a Curve
A curve, \( C \), in \( \mathbb{R}^3 \) can be described by parametric equations of the form
\[
\begin{align*}
x &= x(t) \\
y &= y(t) \\
z &= z(t).
\end{align*}
\]
Any curve can be parameterized in many different ways. For example, the unit circle (traced out once counterclockwise) can be described with the parametric equations
\[
\begin{align*}
x &= \cos(t) \\
y &= \sin(t) \\
0 \leq t \leq 2\pi
\end{align*}
\]
or with the parametric equations
\[
\begin{align*}
x &= \cos(4t) \\
y &= \sin(4t) \\
0 \leq t \leq \frac{\pi}{2}.
\end{align*}
\]
Both of the above examples are parameterizations of the unit circle because in both examples we can see that \( x^2 + y^2 = 1 \) for all \( t \).

If \( t \) denotes time, then
\[
\begin{align*}
x &= x(t) \\
y &= y(t) \\
z &= z(t)
\end{align*}
\]
describes the motion of a particle along the curve \( C \). The location of the particle at time \( t \) is \((x(t), y(t), z(t))\). It is convenient to write the above parametric equations as a single vector equation
\[
r(t) = x(t)i + y(t)j + z(t)k.
\]
The velocity vector of the moving object at time \( t \) is
\[
v(t) = r'(t) = x'(t)i + y'(t)j + z'(t)k.
\]
This vector is always tangent to \( C \) and points in the (instantaneous) direction of motion at time \( t \).

The speed of the particle at time \( t \) is
\[
v(t) = |v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.
\]
(Note that speed is a non–negative scalar.)
The acceleration vector of the particle at time $t$ is

$$a(t) = v'(t) = x''(t)i + y''(t)j + z''(t)k.$$ 

**Example**  Suppose a particle moves along a helical path according to

$$x = \cos(t)$$
$$y = \sin(t)$$
$$z = \frac{1}{2}t$$

where $t$ denotes time. Then the position function is

$$\mathbf{r}(t) = \cos(t)i + \sin(t)j + \frac{1}{2}tk,$$

the velocity function is

$$\mathbf{v}(t) = -\sin(t)i + \cos(t)j + \frac{1}{2}k$$

and the acceleration function is

$$a(t) = -\cos(t)i - \sin(t)j.$$ 

The speed is

$$v(t) = |\mathbf{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}.$$ 

**Example**  Suppose a particle moves along an "involute" path according to

$$x = \cos(t) + t\sin(t)$$
$$y = \sin(t) - t\cos(t)$$

where $t$ denotes time and we assume $t \geq 0$. Then
\[ r(t) = (\cos(t) + t\sin(t))\mathbf{i} + (\sin(t) - t\cos(t))\mathbf{j} \]
\[ v(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} \]
\[ a(t) = (\cos(t) - t\sin(t))\mathbf{i} + (t\cos(t) + \sin(t))\mathbf{j} \]

and the speed is
\[ v(t) = \sqrt{(t\cos(t))^2 + (t\sin(t))^2} = t. \]

The path \( x = \cos(t) + t\sin(t), \ y = \sin(t) - t\cos(t) \) for \( 0 \leq t \leq 1 \)

**Example** Suppose that a particle moves along a parabolic path according to

\[ x = t \]
\[ y = t^2 \]

where \( t \) denotes time. Then

\[ r(t) = t\mathbf{i} + t^2\mathbf{j} \]
\[ v(t) = \mathbf{i} + 2t\mathbf{j} \]
\[ a(t) = 2\mathbf{j} \]

and the speed is
\[ v(t) = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}. \]
The path \( x = t, \ y = t^2 \) for \(-3 \leq t \leq 3\)

**Parameterizing a Curve with Respect to Arc Length**

Suppose we have a curve, \( C \), described by
\[
\begin{align*}
    x &= x(t) \\
    y &= y(t) \\
    z &= z(t).
\end{align*}
\]

Also suppose that \( t \geq t_0 \) (for some given \( t_0 \)) and that \( s(t) \) is the distance (arc length) measured along the curve from the point \((x(t_0), y(t_0), z(t_0))\) to the point \((x(t), y(t), z(t))\). If we think of \( t_0 \) being fixed and \( t \) increasing, then \( ds/dt \) is the rate of change of arc length with respect to \( t \). In particular, if \( t \) denotes time, then \( ds/dt \) is the speed at which the particle is moving. Thus
\[
    \frac{ds}{dt} = v(t)
\]
and
\[
    s(t_0) = 0.
\]

By integration we obtain
\[
    s = \int_{t_0}^{t} v(\tau) \, d\tau.
\]

If this equation can be solved algebraically for \( t \) in terms of \( s \), then we can write \( t \) as a function of \( s \) and reparameterize the curve, \( C \), as
\[
\begin{align*}
    x &= x(t(s)) \\
    y &= y(t(s)) \\
    z &= z(t(s)).
\end{align*}
\]
For any given value of $s$, the point $(x(t(s)), y(t(s)), z(t(s)))$ is at a distance of $s$ (measured along the curve) from the point $(x(t_0), y(t_0), z(t_0))$.

Even if we cannot solve for $t$ in terms of $s$ (algebraically), it is still true that $t$ is an implicitly-defined function of $s$ if we assume that $ds/dt > 0$ for all $t$. We will make this assumption. (Recall that we already know that it must be the case that $ds/dt = v(t) \geq 0$ for all $t$. We making the somewhat stronger assumption that $ds/dt = v(t) > 0$ for all $t$. In terms of motion, this means that the moving object never comes to a momentary stop.) Since $s = s(t)$ is a monotone increasing function of $t$, then it has inverse function, $t = t(s)$. Thus the arc length parameter always exists even if we cannot write it down explicitly. This fact is sufficient for deriving many of the formulas related to motion. In the following three examples we will find arc length parameterizations of curves.

**Example** Let us parameterize the curve

$$
x = \cos(t) \\
y = \sin(t) \\
z = \frac{1}{2} t
$$

with respect to arc length. We will use the base point $(1, 0, 0)$ which corresponds to $t_0 = 0$. Recalling that $v(t) = \sqrt{5}/2$ we have

$$s = \int_0^t \frac{\sqrt{5}}{2} \, dt$$

which, upon computing the integral, becomes

$$s = \frac{\sqrt{5}}{2} t$$

Solving for $t$ gives

$$t = \frac{2\sqrt{5}}{5} s.$$

Substitution into the original parameterization of the curve gives

$$x = \cos\left(\frac{2\sqrt{5}}{5} s\right)$$

$$y = \sin\left(\frac{2\sqrt{5}}{5} s\right)$$

$$z = \frac{\sqrt{5}}{5} s.$$

Observe that if we now let $t = s$ in the above parameterization and say that $t$ denotes time then we obtain
If these are the parametric equations for a particle moving along the helix (with \( t \) denoting time) then

\[
\mathbf{r}(t) = \cos\left(\frac{2\sqrt{5}}{5}t\right)\mathbf{i} + \sin\left(\frac{2\sqrt{5}}{5}t\right)\mathbf{j} + \frac{\sqrt{5}}{5}t\mathbf{k}
\]

\[
\mathbf{v}(t) = -\frac{2\sqrt{5}}{5} \sin\left(\frac{2\sqrt{5}}{5}t\right)\mathbf{i} + \frac{2\sqrt{5}}{5} \cos\left(\frac{2\sqrt{5}}{5}t\right)\mathbf{j} + \frac{\sqrt{5}}{5}\mathbf{k}
\]

and

\[
\mathbf{v}(t) = \sqrt{\left(-\frac{2\sqrt{5}}{5} \sin\left(\frac{2\sqrt{5}}{5}t\right)\right)^2 + \left(\frac{2\sqrt{5}}{5} \cos\left(\frac{2\sqrt{5}}{5}t\right)\right)^2 + \left(\frac{\sqrt{5}}{5}\right)^2}
\]

\[
= \sqrt{\frac{4}{5} \sin^2\left(\frac{2\sqrt{5}}{5}t\right) + \frac{4}{5} \cos^2\left(\frac{2\sqrt{5}}{5}t\right) + \frac{1}{5}}
\]

\[
= 1.
\]

Thus we obtain a parameterization for which the particle is moving along the curve with a constant speed of 1. It will always be the case (in any example) that if \( s \) is an arc length parameter and we then let \( t = s \) and let \( t \) be time, then the equations

\[
x = x(t) \\
y = y(t) \\
z = z(t)
\]

will be the equations of motion for a particle moving along the given path with a constant speed of 1.

**Example** Let us parameterize the curve

\[
x = \cos(t) + t \sin(t) \\
y = \sin(t) - t \cos(t) \\
t \geq 0
\]

with respect to arc length. We will use the base point \((1,0)\) which corresponds to \( t_0 = 0 \). Recall that \( \mathbf{v}(t) = t \) and hence

\[
s = \int_0^t \tau \, d\tau.
\]

This gives
Solving for \( t \) gives

\[ t = \sqrt{2s}. \]

Substitution into the original parameterization of the curve gives

\[
\begin{align*}
x &= \cos(\sqrt{2s}) + \sqrt{2s} \sin(\sqrt{2s}) \\
y &= \sin(\sqrt{2s}) - \sqrt{2s} \cos(\sqrt{2s})
\end{align*}
\]

\( s \geq 0. \)

**Example** Let us parameterize the parabolic curve

\[
\begin{align*}
x &= t \\
y &= t^2
\end{align*}
\]

with respect to arc length. We will use the base point \((0,0)\) which corresponds to \( t_0 = 0 \). Recall that \( v(t) = \sqrt{1 + 4t^2} \) and hence

\[ s = \int_0^t \sqrt{1 + 4t^2} \, dt. \]

It is a challenging exercise (although doable using the integration techniques studied in Calculus II) to find that

\[
\int_0^t \sqrt{1 + 4t^2} \, dt = \frac{1}{4} \ln\left(2t + \sqrt{4t^2 + 1}\right) + \frac{1}{2} t \sqrt{4t^2 + 1}.
\]

Unfortunately we cannot solve the equation

\[ s = \frac{1}{4} \ln\left(2t + \sqrt{4t^2 + 1}\right) + \frac{1}{2} t \sqrt{4t^2 + 1} \]

for \( t \) in terms of \( s \) and hence we cannot explicitly write down an arc length parameterization for the parabola. Nonetheless, the arc length parameter still exists because \( s \) is an increasing function of \( t \). The function \( t(s) \) is defined implicitly by the above equation. Fortunately the only fact that we will need in order to derive many of the aspects of motion for this example is the fact that

\[ \frac{ds}{dt} = \sqrt{1 + 4t^2}. \]

(The same is true for other examples in which we are not able to write down the function \( t(s) \) explicitly: \( ds/dt = v(t) \) is all we need to be able to deduce a lot of information about the motion.)

**Curvature**

For a curve

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \]

we define the unit tangent vector, \( \mathbf{T}(t) \), at a point on the curve corresponding to
parameter value \( t \) to be the unit vector that points in the same direction as the velocity vector. That is
\[
\mathbf{T}(t) = \frac{1}{|\mathbf{v}(t)|} \mathbf{v}(t) = \frac{1}{v(t)} \mathbf{v}(t).
\]
If we reparameterize this curve in terms of arc length (setting \( t = t(s) \)), then \( d\mathbf{T}/ds \) tells us the instantaneous rate at which the unit tangent vector is changing with respect to arc length. Note that \( \mathbf{T} \) always has length one and hence its length does not change. It can only change direction. Thus \( d\mathbf{T}/ds \) tells us the rate at which the direction of the curve is changing with respect to distance travelled along the curve. The curvature function, \( \kappa \), of the curve is defined to be the magnitude of this rate of change. Thus we define
\[
\kappa(t) = \left| \frac{d\mathbf{T}}{ds} \right|.
\]

As we have seen, it can be difficult or impossible to find the arc length parameter \( s \) explicitly and thus we might not be able to use the above definition of curvature directly to compute curvatures. Fortunately all we need is the fact that \( ds/dt = v(t) \) because by the Chain Rule
\[
\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = v(t) \frac{d\mathbf{T}}{ds}
\]
and, by taking magnitudes and recalling that we are assuming that \( v(t) > 0 \) for all \( t \), we get
\[
\left| \frac{d\mathbf{T}}{dt} \right| = v(t) \left| \frac{d\mathbf{T}}{ds} \right|
\]
and this gives
\[
\kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)}.
\]
Hence we can compute curvature using any given parameterization of a curve. It is not necessary to know the arc length parameterization.

**Exercise** Show that a circle of radius \( a > 0 \) has curvature \( 1/a \) at all points on the circle. Hint: Parametric equations for the circle are \( x = a \cos(t) \), \( y = a \sin(t) \).

**Example** Let us find the curvature function for the helical curve
\[
x = a \cos(t) \\
y = a \sin(t) \\
z = bt
\]
where \( a > 0 \) and \( b > 0 \) are given constants. For this curve we have
\[
\mathbf{r}(t) = a \cos(t) \mathbf{i} + a \sin(t) \mathbf{j} + bt \mathbf{k}
\]
\[
\mathbf{v}(t) = -a \sin(t) \mathbf{i} + a \cos(t) \mathbf{j} + b \mathbf{k}
\]
and
\[ v(t) = \sqrt{a^2 + b^2}. \]

Also
\[ T(t) = \frac{1}{v(t)} v(t) = \frac{1}{\sqrt{a^2 + b^2}} (-a \sin(t)i + a \cos(t)j + bk). \]

Thus
\[ T'(t) = \frac{1}{\sqrt{a^2 + b^2}} (-a \cos(t)i - a \sin(t)j) \]

and
\[ |T'(t)| = \frac{a}{\sqrt{a^2 + b^2}}. \]

The curvature function (as a function of \( t \)) is thus
\[ \kappa(t) = \frac{|T'(t)|}{v(t)} = \frac{a}{a^2 + b^2}. \]

Note that if \( b \approx 0 \), then \( \kappa(t) \approx 1/a \). This makes sense because in this case the helix is very "tightly wound" and hence the curvature is approximately the same as the curvature of a circle of radius \( a \). Likewise if \( b \) is a very large number, then \( \kappa(t) \approx 0 \). This also makes sense because if \( b \) is very large then the helix is very “stretched out” and thus has very small curvature.

**Example**  Let us fund the curvature for the involute curve
\[ x = \cos(t) + t \sin(t) \]
\[ y = \sin(t) - t \cos(t) \]
\[ t \geq 0. \]

Here we have
\[ v(t) = t \cos(t)i + t \sin(t)j \]

and \( v(t) = t \). Also
\[ T(t) = \frac{1}{t} (t \cos(t)i + t \sin(t)j) = \cos(t)i + \sin(t)j \]

and
\[ T'(t) = -\sin(t)i + \cos(t)j \]

and thus \( |T'(t)| = 1 \). The curvature (as a function of \( t \)) is thus
\[ \kappa(t) = \frac{|T'(t)|}{v(t)} = \frac{1}{t}. \]

This makes intuitive sense because the involute curves becomes less and less curvy as we travel along it (with increasing \( t \)). Notice that \( \kappa(t) \to 0 \) as \( t \to \infty \) and \( \kappa(t) \to \infty \) as \( t \to 0^+ \).

**Exercise**  Show that if \( y = f(x) \) is any twice–differentiable function, then the
The curvature function of the graph of this function is
\[ \kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}. \]

**Hint:** The graph of \( y = f(x) \) can be parameterized as
\[
\begin{align*}
  x &= x \\
  y &= f(x).
\end{align*}
\]

**Exercise** Use the result of the above exercise to show that the curvature function for the graph of \( y = x^2 \) is
\[ \kappa(x) = \frac{2}{(1 + 4x^2)^{3/2}}. \]

It can thus be seen that the graph of \( y = x^2 \) has maximum curvature at the vertex of the parabola (where \( x = 0 \)) and that the curvature becomes smaller as \( x \to \infty \) and as \( x \to -\infty \). It is also seen that \( \kappa(-x) = \kappa(x) \) for all \( x \) (i.e., \( \kappa \) is an even function of \( x \)). All of these observations are in accord with what we expect given our previous knowledge of the shape of the parabola.

**Tangential and Normal Components of Acceleration**

Acceleration is the rate of change of velocity with respect to time. Since velocity is a vector, it has both a magnitude (which is the instantaneous speed of the particle at a given moment of time) and a direction (which is the instantaneous direction in which the particle is travelling at a given moment of time). What is the contribution of each of these two components of acceleration to the acceleration vector? We will see.

To begin, note that the velocity vector is
\[ \mathbf{v}(t) = \mathbf{v}(t) \left( -\frac{1}{\mathbf{v}(t)} \mathbf{v}(t) \right) = \mathbf{v}(t) \mathbf{T}(t). \]

Thus the acceleration vector is
\[ \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{v}(t) \mathbf{T}'(t) + \mathbf{v}'(t) \mathbf{T}(t). \]

Since \( |\mathbf{T}(t)| = 1 \) for all \( t \), then at all times \( t \) we have
\[ \mathbf{T}(t) \cdot \mathbf{T}(t) = |\mathbf{T}(t)|^2 = 1. \]

Differentiation with respect to \( t \) gives
\[ \frac{d}{dt} (\mathbf{T}(t) \cdot \mathbf{T}(t)) = \frac{d}{dt} (1) \]
for all \( t \) which (by using the Product Rule of differentiation for vector–valued functions) gives
\[ \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0 \]
for all \( t \). In other words \( \mathbf{T}'(t) \) is orthogonal to \( \mathbf{T}(t) \) at all times \( t \). In addition, since \( \mathbf{T}'(t) \) is the rate of change of \( \mathbf{T}(t) \), then \( \mathbf{T}'(t) \) points in the general direction that the curve is
turning (called the concave side of the curve). For convenience, we will introduce the unit vector that points in the same direction as $T'(t)$. This vector, which is called the unit normal vector, is

$$N(t) = \frac{1}{|T'(t)|} T'(t)$$

Unit Tangent and Normal Vectors

We can thus write the acceleration vector as

$$a(t) = v(t)|T'(t)|N(t) + v'(t)T(t).$$

Recall that

$$\kappa(t) = \frac{|T'(t)|}{v(t)}.$$ 

Thus $|T'(t)| = v(t)\kappa(t)$ and we see that we can write the acceleration vector as

$$a(t) = \kappa(t)v(t)^2 N(t) + v'(t)T(t).$$

The quantity

$$a_T(t) = v'(t)$$

is called the tangential component of acceleration. It tells us the rate of change of speed of the moving particle. It is a force acting in the instantaneous direction of motion.

The quantity

$$a_N(t) = \kappa(t)v(t)^2$$

is called the normal component of acceleration. It tells us the rate of change of direction of the moving particle. It is a force that acts perpendicular to the path of motion and in the direction in which turning is taking place. Not surprisingly, curvature
is involved in the normal component of acceleration and so is speed.

The scalar component of acceleration is usually easy to compute but the normal component involves curvature (which can be tedious to compute). Fortunately we can compute \( a_N(t) \) without first computing the curvature. To see how this can be done, recall the equation

\[
|a + b|^2 = |a|^2 + |b|^2 + 2a \cdot b
\]

which holds for any two vector \( a \) and \( b \). When we apply this to the equation

\[
a(t) = a_N(t)N(t) + a_T(t)T(t),
\]

we obtain

\[
|a(t)|^2 = |a_N(t)N(t) + a_T(t)T(t)|^2
\]

\[
= (a_N(t))^2|N(t)|^2 + (a_T(t))^2|T(t)|^2 + 2a_N(t)a_T(t)(N(t) \cdot T(t)).
\]

However, since \( N(t) \) and \( T(t) \) both have magnitude 1 and since they are also orthogonal to each other, the above equation simplifies to

\[
|a(t)|^2 = (a_N(t))^2 + (a_T(t))^2.
\]

The normal component of acceleration is thus given by

\[
a_N(t) = \sqrt{|a(t)|^2 - (a_T(t))^2}.
\]

Example

Suppose that a particle travels along the graph of \( y = \sin(x) \) according to

\[
x = t
\]

\[
y = \sin(t)
\]

(where \( t \) is time). Then

\[
r(t) = ti + \sin(t)j
\]

\[
v(t) = i + \cos(t)j
\]

and the speed is

\[
v(t) = \sqrt{1 + \cos^2(t)}.
\]

Notice that the speed is always some value between 1 and \( \sqrt{2} \) because \( \cos^2(t) \) always has a value between 0 and 1. The speed of this particle varies with time but it is never going slower than 1 or faster than \( \sqrt{2} \). The acceleration is

\[
a(t) = -\sin(t)j
\]

and thus \( |a(t)| = |\sin(t)| \). The tangential component of acceleration is

\[
a_T(t) = v'(t) = \frac{1}{2\sqrt{1 + \cos^2(t)}}(-2\cos(t)\sin(t)) = \frac{-\sin(t)\cos(t)}{\sqrt{1 + \cos^2(t)}}
\]

and the normal component of acceleration is
Below we show the graphs of $\gamma = \sin(t)$ (in black), $a_T(t)$ (in red) and $a_N(t)$ (in green).

By looking at these graphs we can see what is happening if we imagine a car driving along the roadway $y = \sin(x)$. For instance, when $t = 0$, the car is at the point $(0,0)$. Between times $t = 0$ and $t = \pi/2$, the scalar acceleration ($a_T(t)$) is negative which means the car’s speed is decreasing during that time interval. (Good thing because it is coming to a sharp curve in the road.) At time $t = \pi/2$, the sharp curve at the point $(\pi/2, 1)$ is reached. After rounding the curve, the scalar acceleration becomes positive meaning that the car begins speeding up. At time $t = \pi$ it begins slowing down again. (Good thing because another sharp curve coming up at the point $(3\pi/2, -1)$. Notice that the normal component of acceleration is always positive. It will always be the case (no matter which example we study) that $a_N(t) = \kappa(t)v(t)^2 \geq 0$ because $\kappa(t)$ is always non–negative. In this example, $a_N(t)$ is largest at the sharp curves in the road because $\kappa(t)$ is largest there (even though in fact $v(t)$ is smallest.

\[
a_N(t) = \sqrt{|a(t)|^2 - (a_T(t))^2}
= \sqrt{\sin(t)^2 - \left(\frac{-\sin(t)\cos(t)}{\sqrt{1 + \cos^2(t)}}\right)^2}
= \sqrt{\sin^2(t) - \frac{\sin^2(t)\cos^2(t)}{1 + \cos^2(t)}}
= \sqrt{\frac{\sin^2(t)(1 + \cos^2(t))}{1 + \cos^2(t)} - \frac{\sin^2(t)\cos^2(t)}{1 + \cos^2(t)}}
= \frac{\sin(t)}{\sqrt{1 + \cos^2(t)}}.
\]
there). It is not always true (in other examples we might study) that \( a_N(t) \) is largest at the sharpest curves in the road. We can envision a car travelling over a gently curving road at a very fast speed. Even though the road is gently curving, the car experiences a large change in direction over a short period of time because it is travelling very fast. Then when it reaches a sharp curve it might (hopefully) slow way down so that \( \kappa(t)v(t)^2 \) is not too large even though \( \kappa(t) \) is large.