MATH 2203 (Calculus III) - Quiz 1 Solutions
January 21, 2015
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Instructions. This is a take–home quiz. It is due to be handed in to me by Monday, January 26 at class time. You may work on this quiz alone or in a group of one or two other people. (If you work in a group, then you will hand in only one paper and all in the group will receive the same grade on the quiz.) You may use any books or other resources that you need to do the quiz with the exception of consulting other people. Your solutions must include sufficient detail so that I am able to understand your reasoning process in solving the problems. The paper that you hand in must be written neatly and you must use correct mathematical notation and writing. Write in complete sentences! Do not hand in something that looks like your scratch paper. Writing, notation and neatness will taken into account in my grading of the quiz. All papers must be stapled together (not paper–clipped or folded). Points will be deducted for no staple.

1. Find the center and the radius of the sphere described by the equation
   \[ x^2 + y^2 + z^2 - 4x + 2y + 6z + 10 = 0. \]

   Solution: We first rewrite this equation as
   \[ x^2 - 4x + y^2 + 2y + z^2 + 6z = -10 \]
   and then use the method of completing the square to rewrite it again as
   \[ x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 + 6z + 9 = -10 + 4 + 1 + 9 \]
   or as
   \[ (x - 2)^2 + (y + 1)^2 + (z + 3)^2 = 4 \]
   or as
   \[ (x - 2)^2 + (y - (-1))^2 + (z - (-3))^2 = 2^2 \]
   Having written the equation in the above form, we can now see that this sphere has center at the point \((2, -1, -3)\) and that the radius of this sphere is 2.

2. Let \(u\) and \(v\) be the vectors \(u = \langle -6, -7 \rangle\) and \(v = \langle 9, -2 \rangle\).
   a. Draw pictures of the standard representatives of \(u\) and \(v\).
b. Find the vector $\mathbf{u} + \mathbf{v}$.

Solution: $\mathbf{u} + \mathbf{v} = \langle -6 + 9, -7 + (-2) \rangle = \langle 3, -9 \rangle$.

c. Draw a picture that illustrates the vector addition $\mathbf{u} + \mathbf{v}$. Your picture should include the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$. (Not all three of these vectors will be drawn in standard position though.)
d. Draw a picture of the standard representative of the vector $-u$.
Answer: $-u = \langle 6, 7 \rangle$. (I have omitted the picture.)

e. Circle the correct choices: The length of the vector $-u$ is (greater than/less than/the same as) the length of the vector $u$ and the vector $-u$ points in (the same/the opposite) direction of the vector $u$.

3. An airplane flying in the direction $30^\circ$ north of west at 400 miles per hour encounters an 87 mile per hour headwind blowing directly east. The airplane holds it compass heading $30^\circ$ north of west but, because of the headwind, acquires a new ground speed and direction. What are they? (Show you work in detail. Include diagrams, all relevant calculations, and narrative explaining your reasoning.)

Solution: The air velocity vector of the plane is
$$V = 400 \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle = \langle -200\sqrt{3}, 200 \rangle.$$ The wind velocity vector is
$$W = 87 \langle 1, 0 \rangle = \langle 87, 0 \rangle.$$ The newly-acquired ground velocity vector is
$$G = V + W = \langle -200\sqrt{3} + 87, 200 \rangle.$$ See the picture below:
The ground speed of the plane is thus

$$|G| = \sqrt{(-200\sqrt{3} + 87)^2 + 200^2} \approx 327.56 \text{ miles per hour.}$$

In the picture given above, the true direction of the plane is $30^\circ + \theta$ north of west. We see that

$$\tan(30^\circ + \theta) = \frac{200}{200\sqrt{3} - 87}$$

and thus

$$30^\circ + \theta = \arctan\left(\frac{200}{200\sqrt{3} - 87}\right) \approx 37.63^\circ.$$ 

In conclusion the true speed (ground speed) of the plane is about 327.56 miles per hour and the true direction is about $37.63^\circ$ north of west.

4. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors $\mathbf{u} = (-6,-7)$ and $\mathbf{v} = (9,-2)$.
   a. Compute the dot product, $\mathbf{u} \cdot \mathbf{v}$.
      \textbf{Solution:} $\mathbf{u} \cdot \mathbf{v} = (-6)(9) + (-7)(-2) = -40$.
   b. Based on what you found in part a, determine whether the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is acute, obtuse, or a right angle. Explain your answer.
      \textbf{Answer:} The angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is obtuse because $\mathbf{u} \cdot \mathbf{v} < 0$. 

c. Find $|u|$ and $|v|$.

**Solution:**

\[ |u| = \sqrt{(-6)^2 + (-7)^2} = \sqrt{85} \]

and

\[ |v| = \sqrt{9^2 + (-2)^2} = \sqrt{85}. \]

d. Find the angle between $u$ and $v$.

**Solution:** If $\theta$ is the angle between $u$ and $v$, then

\[ \cos(\theta) = \frac{u \cdot v}{|u||v|} = \frac{-40}{85} = -\frac{8}{17}. \]

Therefore

\[ \theta = \arccos\left( -\frac{8}{17} \right) \approx 118.07^\circ. \]

5. Consider the constant force vector $F = 10\mathbf{i} - 20\mathbf{j}$.

a. Find the magnitude and the direction of $F$. (Recall that the magnitude of $F$ is $|F|$ and that we have defined the direction of $F$ to be the unit vector that points in the same direction of $F$.) Write $F$ in the form

\[ F = |F|(\text{direction of } F). \]

**Solution:** The magnitude of $F$ is

\[ |F| = \sqrt{(10)^2 + (-20)^2} = 10\sqrt{5}. \]

The direction of $F$ is

\[ \frac{1}{|F|}F = \frac{1}{10\sqrt{5}}(10\mathbf{i} - 20\mathbf{j}) = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}. \]

Thus we can write $F$ as

\[ F = 10\sqrt{5}\left(\frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}\right) \]

b. Find the work done by $F$ in moving an object from the point $P(-4, 3)$ along a straight line path to the point $Q(3, 2)$. Assume that distances are measured in meters.

**Solution:** Note that the vector pointing from $P$ to $Q$ is

\[ \overrightarrow{PQ} = (3 - (-4))\mathbf{i} + (2 - 3)\mathbf{j} = 7\mathbf{i} - \mathbf{j}. \]

The work done by $F$ in moving an object from the point $P$ along a straight line path to the point $Q$ is

\[ F \cdot \overrightarrow{PQ} = (10\mathbf{i} - 20\mathbf{j}) \cdot (7\mathbf{i} - \mathbf{j}) = (10)(7) + (-20)(-1) = 90 \text{ Newton–meters} = 90 \text{ Joules}. \]