1. The position vector, at time $t$, of a particle in motion in the $xy$ plane is given by
   \[ r(t) = -5 \cos(t)i - 5 \sin(t)j. \]

   a. Find the path of motion of the particle. (Explain and draw a picture.)
   b. Find the velocity vector, $v(t)$, at any time $t$.
   c. Find the speed at any time $t$.
   d. Find the acceleration vector, $a(t)$, at any time $t$.
   e. In the picture that you drew in part a, draw the position vector and the velocity vector at time $t = 0$. Do the same at time $t = \frac{2\pi}{3}$.
   f. Is the particle travelling in a clockwise or counterclockwise direction?

   **Solution:** Note that when we write the position function in parametric form, we obtain
   \[ x = -5 \cos(t) \]
   \[ y = -5 \sin(t). \]
   Since $x^2 + y^2 = 25$ at all times $t$, we see that the path of motion is a circle of radius 5 centered at $(0,0)$. The velocity vector is
   \[ v(t) = 5 \sin(t)i - 5 \cos(t)j \]
   and the speed is
   \[ |v(t)| = \sqrt{(5 \sin(t))^2 + (-5 \cos(t))^2} = 5. \]
   The acceleration vector is
   \[ a(t) = 5 \cos(t)i + 5 \sin(t)j. \]
   The position and velocity vectors at time $t = 0$ and $t = \frac{2\pi}{3}$ are as follows:
\[ \mathbf{r}(0) = -5 \cos(0) \mathbf{i} - 5 \sin(0) \mathbf{j} = -5 \mathbf{i} \]
\[ \mathbf{v}(0) = 5 \sin(0) \mathbf{i} - 5 \cos(0) \mathbf{j} = -5 \mathbf{j} \]
\[ \mathbf{r}\left(\frac{2\pi}{3}\right) = -5 \cos\left(\frac{2\pi}{3}\right) \mathbf{i} - 5 \sin\left(\frac{2\pi}{3}\right) \mathbf{j} = \frac{5}{2} \mathbf{i} - \frac{5\sqrt{3}}{2} \mathbf{j} \]
\[ \mathbf{v}\left(\frac{2\pi}{3}\right) = 5 \sin\left(\frac{2\pi}{3}\right) \mathbf{i} - 5 \cos\left(\frac{2\pi}{3}\right) \mathbf{j} = \frac{5\sqrt{3}}{2} \mathbf{i} + \frac{5}{2} \mathbf{j}. \]

This is illustrated below.
It can be seen from the illustration that the direction of motion is counterclockwise. Recall that the velocity vector is always tangent to the path of motion and points in the direction of motion.

2. The velocity vector, at time $t$, of a particle is given by

$$v(t) = ti - 2tj$$

and the particle is located at the point $(-2, 0)$ at time $t = 0$.

a. Find the position vector of the particle at any time $t$. 
b. Sketch the path of motion of the particle over the time interval $0 \leq t \leq 5$. (You can use Mathematica to do this if you want to or do it by hand on graph paper.)

c. At what point is the particle located at time $t = 1$?

d. Will the particle ever arrive at the point $(6, -14)$? If so, at what time will this happen?

**Solution:** The position vector is

$$\mathbf{r}(t) = \frac{1}{2} t^2 \mathbf{i} - t^2 \mathbf{j} + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector. At time $t = 0$ we have

$$\mathbf{r}(0) = \mathbf{C} = -2\mathbf{i}.$$  

Thus

$$\mathbf{r}(t) = \left( \frac{1}{2} t^2 - 2 \right) \mathbf{i} - t^2 \mathbf{j}.$$  

Parametric equations for the path of motion of the particle are

$$x = \frac{1}{2} t^2 - 2$$

$$y = -t^2.$$  

By solving each of these equations for $t^2$ we get

$$t^2 = 2x + 4$$

$$t^2 = -y.$$  

Thus the path of motion is the line $y = -2x - 4$. At time $t = 1$, the particle is located at the point $(-3/2, -1)$. It will never arrive at the point $(6, -14)$ because the line $y = -2x - 4$ does not pass through that point. The path of motion over the time interval $0 \leq t \leq 5$ is shown below.
3. A man is standing 20 feet away from a box of width 5 feet and height 3 feet (See
Diagram) and is tossing bean bags – trying to get them to land in the box. He is tossing the bean bags from an initial height of 3 feet and with an initial angle of $\alpha = 45^\circ$. What range of initial speeds ($v_0$) must the man use to ensure that the bean bags land in the box? (Assume that the only vertical force that acts on the bean bags during their flight is the acceleration due to gravity. Thus assume that $a(t) = -32\mathbf{j}$.) Be detailed in your solution. Write in sentences.

**Solution:** Since $a(t) = -32\mathbf{j}$, we have

$$v(t) = -32t\mathbf{j} + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector. Also, if the initial speed is $v_0$, then the initial velocity vector is

$$v_0 = v_0 \cos(45^\circ)\mathbf{i} + v_0 \sin(45^\circ)\mathbf{j} = \frac{\sqrt{2}v_0}{2}\mathbf{i} + \frac{\sqrt{2}v_0}{2}\mathbf{j}.$$ 

Since

$$v(0) = \mathbf{C} = \frac{\sqrt{2}v_0}{2}\mathbf{i} + \frac{\sqrt{2}v_0}{2}\mathbf{j},$$

we have

$$v(t) = \frac{\sqrt{2}v_0}{2}\mathbf{i} + \left(\frac{\sqrt{2}v_0}{2} - 32t\right)\mathbf{j}.$$ 

By antidifferentiation, we find that the position vector at time $t$ is

$$r(t) = \frac{\sqrt{2}v_0}{2}t\mathbf{i} + \left(\frac{\sqrt{2}v_0}{2}t - 16t^2\right)\mathbf{j} + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector. Since

$$r(0) = \mathbf{C} = 3\mathbf{j},$$

we have

$$r(t) = \frac{\sqrt{2}v_0}{2}t\mathbf{i} + \left(\frac{\sqrt{2}v_0}{2}t - 16t^2 + 3\right)\mathbf{j}.$$ 

Parametric equations for the path of motion are

$$x(t) = \frac{\sqrt{2}v_0}{2}t$$

$$y(t) = \frac{\sqrt{2}v_0}{2}t - 16t^2 + 3.$$
In order for the bean bag to get into the box, we must have values of \( t \) and \( v_0 \) such that

\[
20 \leq x(t) \leq 25
\]

and

\[
y(t) = 3.
\]

The equation \( y(t) = 3 \) is the same as

\[
\frac{\sqrt{2} v_0}{2} - 16t^2 = 0
\]

and this equation holds when \( t = 0 \) (which is when the bean bag is first tossed) and when

\[
\frac{\sqrt{2} v_0}{2} = 16t
\]

or

\[
t = \frac{\sqrt{2} v_0}{32}.
\]

At this value of \( t \) we have

\[
x(t) = \frac{\sqrt{2} v_0}{2} \left( \frac{\sqrt{2} v_0}{32} \right) = \frac{v_0^2}{32}
\]

and we also want to have \( 20 \leq x(t) \leq 25 \) which means that we must have

\[
20 \leq \frac{v_0^2}{32} \leq 25.
\]

This equation can be written as

\[
640 \leq v_0^2 \leq 800
\]

or

\[
\sqrt{640} \leq v_0 \leq \sqrt{800}.
\]

We conclude that the bean bag will land in the box if the initial tossing speed is in the (approximate) range of 25.3 ft/sec to 28.28 ft/sec.

4. A man is standing 20 feet away from a box of width 5 feet and height 3 feet (See Diagram) and is tossing bean bags – trying to get them to land in the box. He is tossing the bean bags from an initial height of 3 feet and with an initial speed of \( v_0 = 40 \) feet per second. What ranges of angles \( \alpha \) must the man use to ensure that the bean bags land in the box? (Assume that the only vertical force that acts on the bean bags during their flight is the acceleration due to gravity. Thus assume that \( a(t) = -32j \).) Be detailed in your solution. Write in sentences.

Solution: Since \( a(t) = -32j \), we have

\[
v(t) = -32tj + C
\]

where \( C \) is a constant vector. Also, if the initial tossing angle is \( \alpha \), then the initial velocity vector is

\[
v_0 = 40 \cos(\alpha)i + 40 \sin(\alpha)j.
\]

Since

\[
v(0) = C = 40 \cos(\alpha)i + 40 \sin(\alpha)j,
\]

we have
\[ \mathbf{v}(t) = 40 \cos(\alpha) \mathbf{i} + (40 \sin(\alpha) - 32t) \mathbf{j}. \]

The position vector at time \( t \) is
\[ \mathbf{r}(t) = 40 \cos(\alpha) \mathbf{i} + (40 \sin(\alpha) t - 16t^2) \mathbf{j} + \mathbf{C} \]
where \( \mathbf{C} \) is a constant vector.

Since
\[ \mathbf{r}(0) = \mathbf{C} = 3 \mathbf{j}, \]
we have
\[ \mathbf{r}(t) = 40 \cos(\alpha) \mathbf{i} + (40 \sin(\alpha) t - 16t^2 + 3) \mathbf{j}. \]

Parametric equations for the path of motion are
\[ x(t) = 40 \cos(\alpha) t \]
\[ y(t) = 40 \sin(\alpha) t - 16t^2 + 3. \]

In order for the bean bag to get into the box, we must have values of \( t \) and \( \alpha \) such that
\[ 20 \leq x(t) \leq 25 \]
and
\[ y(t) = 3. \]

The equation \( y(t) = 3 \) is the same as
\[ 40 \sin(\alpha) t - 16t^2 = 0 \]
and this equation holds when \( t = 0 \) (which is when the bean bag is first tossed) and when
\[ 40 \sin(\alpha) = 16t \]
or
\[ t = \frac{5}{2} \sin(\alpha). \]

At this value of \( t \) we have
\[ x(t) = 40 \cos(\alpha) \left( \frac{5}{2} \sin(\alpha) \right) = 100 \sin(\alpha) \cos(\alpha) = 50 \sin(2\alpha). \]
and we also want to have \( 20 \leq x(t) \leq 25 \) which means that we must have
\[ 20 \leq 50 \sin(2\alpha) \leq 25. \]

This equation can be written as
\[ \frac{2}{5} \leq \sin(2\alpha) \leq \frac{1}{2}. \]

Here is a graph of \( y = \sin(2\alpha) \) along with the horizontal lines \( y = 2/5 \) and \( y = 1/2 \). There are four points of intersection and these will give us the critical values of \( \alpha \) that we need.
Solving \( \sin(\alpha) = 2/5 \) for \( 0^\circ \leq \alpha \leq 90^\circ \) (which is equivalent to \( 0^\circ \leq 2\alpha \leq 180^\circ \) ) gives

\[ 2\alpha = \arcsin(2/5) \]

or

\[ 2\alpha = 180^\circ - \arcsin(2/5) \]

which is equivalent to

\[ \alpha = \frac{1}{2} \arcsin(2/5) \approx 11.79^\circ \]

or

\[ \alpha = 90^\circ - \frac{1}{2} \arcsin(2/5) \approx 78.21^\circ. \]

Solving \( \sin(2\alpha) = 1/2 \) for \( 0^\circ \leq \alpha \leq 90^\circ \) (which is equivalent to \( 0^\circ \leq 2\alpha \leq 180^\circ \) ) gives

\[ 2\alpha = \arcsin(1/2) = 30^\circ \]

or

\[ 2\alpha = 180^\circ - \arcsin(1/2) = 150^\circ \]

which is equivalent to

\[ \alpha = 15^\circ \]

or

\[ \alpha = 75^\circ. \]

We conclude that the bean bag will land in the box if the tossing angle is either in the range \( 11.79^\circ \leq \alpha \leq 15^\circ \) or in the range \( 75^\circ \leq \alpha \leq 78.21^\circ \).