1. For the function \( f(x,y) = (4x - x^2)(4y - y^2) \),
   a. Complete the following table for \( f \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( f(x,y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1.7</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

   b. Use Mathematica to plot a graph of the function \( f(x,y) = (4x - x^2)(4y - y^2) \) over the domain \( D = [0, 4] \times [0, 4] \). (Use `Plot3D`.)

   c. Based on the what you did in parts a and b, at what point in \((x,y) \in D \) does it appear that \( f \) achieves it maximum value? What is the maximum value.

   d. Use Mathematica to plot the level curves (contours) of \( f \) (Use `ContourPlot`).

   **Solution:** We use `Plot3D[(4 x - x^2)*(4 y - y^2), {x, 0, 4}, {y, 0, 4}]` and obtain the following picture.
We use ContourPlot[(4 x - x^2)*(4 y - y^2), {x, 0, 4}, {y, 0, 4}] and obtain the following contour plot.
It appears that $f$ achieves its maximum value at $(2,2)$. The value is $f(2,2) = 16$.

2. For the same function, $f$, given in Question 1,
   a. Compute the partial derivative, $f_x$, and then compute $f_x(3,1)$.
   b. Compute the partial derivative, $f_y$, and then compute $f_y(3,1)$.
   c. On the graph of $f$, draw small tangent lines at the point $(3,1,9)$ to illustrate the results that you found in parts a and b.
   d. Do the results that you obtained in part a, b, and c makes sense to you in regard to the graph of $f$ at the point $(3,1,9)$? Explain why or why not.

**Solution:** We have

\[
    f_x = (4 - 2x)(4y - y^2)
\]

\[
    f_y = (4x - x^2)(4 - 2y).
\]

Also

\[
    f_x(3,1) = (4 - 2(3))(4(1) - 1^2) = -6
\]

\[
    f_y(3,1) = (4(3) - 3^2)(4 - 2(1)) = 6.
\]

The value of $f_x(3,1)$ is the slope of the tangent line to the graph of $z = f(x,1)$ at the point $(3,1,9)$. The value of $f_y(3,1)$ is the slope of the tangent line to the graph of $z = f(3,y)$ at the point $(3,1,9)$. These tangent lines are illustrated below and they do make sense because we can see in the picture that $f$ has negative slope in the $x$ direction at the point $(3,1,9)$ and has positive slope in the $y$ direction at $(3,1,9)$.
3. For the same function $f$ given in Questions 1 and 2:
   a. Compute $f_{xx}$ and $f_{yy}$ and then compute $f_{xx}(3, 1)$ and $f_{yy}(3, 1)$
   b. Explain what the values of $f_{xx}(3, 1)$ and $f_{yy}(3, 1)$ tell you about the graph of $f$ at the point $(3, 1, 9)$.
   c. Compute $f_{xy}$ and $f_{yx}$.
   d. You should have found in part c that $f_{xy} = f_{yx}$. Is this unusual? Explain.

**Answers:** In Question 2 we found

$$f_x = (4 - 2x)(4y - y^2)$$
$$f_y = (4x - x^2)(4 - 2y).$$

From this we obtain

$$f_{xx} = -2(4y - y^2)$$
$$f_{yy} = -2(4x - x^2)$$

and

$$f_{xx}(3, 1) = -2(4(1) - 1^2) = -6$$
$$f_{yy}(3, 1) = -2(4(3) - 3^2) = -6.$$  

These values, since they are both negative, tell us that the graph of $f$ is concave down in both the $x$ direction and the $y$ direction at the point $(3, 1, 9)$. This can be seen to be correct by looking at the picture given in Question 3.

Also

$$f_{xy} = (4 - 2x)(4 - 2y)$$
$$f_{yx} = (4 - 2x)(4 - 2y).$$

It turns out that $f_{xy} = f_{yx}$ as expected because $f_{xy}$ and $f_{yx}$ are both continuous functions and
thus Clairaut’s Theorem guarantees that $f_{xy} = f_{yx}$.

4. Suppose that

$$
\begin{align*}
\quad & w = f(x, y) \\
\quad & x = g(r, s) \\
\quad & y = h(r, s) \\
\quad & r = m(t) \\
\quad & s = n(t).
\end{align*}
$$

a. Draw a chain diagram that illustrates the connections between the variables $w, x, y, r, s$ and $t$ and then write down a Chain Rule for computing $\partial w / \partial t$.

b. Use the Chain Rule that you wrote down in part a to compute $\partial w / \partial t$ in the case that

$$
\begin{align*}
\quad & w = x^2 - 2\cos(xy) \\
\quad & x = r^2 - 2s \\
\quad & y = e^s \\
\quad & r = t^2 \\
\quad & s = 3t^3.
\end{align*}
$$

c. For the situation in part b, find $|\partial w / \partial t|_{t=1}$.

**Solution:** The Chain Diagram is

![Chain Diagram](image)

and the applicable Chain Rule is

$$
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} \frac{\partial s}{\partial t}
$$

$$
+ \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \frac{\partial s}{\partial t}.
$$

For the situation in part b we have
\[
\frac{\partial w}{\partial t} = (2x + 2y \sin(xy))(2r)(2t) + (2x + 2y \sin(xy))(-2)(9t^2) \\
+ (2x \sin(xy))(0)(2t) + (2x \sin(xy))(e^r)(9t^2) \\
= (2x + 2y \sin(xy))(4rt - 18t^2) + 18xt^2e^s \sin(xy).
\]

When \( t = 1 \), we have

\[
w = 25 - 2 \cos(-5e^3) \\
x = -5 \\
y = e^3 \\
r = 1 \\
s = 3
\]

and we see that

\[
\left. \left| \frac{\partial w}{\partial t} \right| \right|_{t=1} = (2(-5) + 2e^3 \sin(-5e^3))(4 - 18) + 18(-5)e^3 \sin(-5e^3) \\
\approx -104.35.
\]