1. Let $C$ be the segment of the parabola $y = x^2 + 2x$ in $\mathbb{R}^2$ with endpoints at $(-3, 3)$ and $(2, 8)$. (A picture of $C$ is given below.) Let 

$$f(x, y) = y - x^2 + 2.$$ 

Evaluate the line integral 

$$\int_C f(x, y) \, ds.$$

**Solution:** The parabolic segment $C$ can be parameterized by 

$$x = t,$$

$$y = t^2 + 2t,$$

$$-3 \leq t \leq 2.$$ 

In vector form, this is 

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2t)\mathbf{j}.$$ 

We thus have 

$$\mathbf{r}'(t) = \mathbf{i} + (2t + 2)\mathbf{j}.$$
and \(|r'(t)| = \sqrt{1^2 + (2t + 2)^2} = \sqrt{4t^2 + 8t + 5}\). Therefore

$$
\int_C f(x,y) \, ds = \int_C (y - x^2 + 2) \, ds \\
= \int_{-3}^{2} (t^2 + 2t - t^2 + 2) \sqrt{4t^2 + 8t + 5} \, dt \\
= \int_{-3}^{2} (2t + 2) \sqrt{4t^2 + 8t + 5} \, dt.
$$

To evaluate this integral, we use the simple substitution \(u = 4t^2 + 8t + 5\) which gives \(du = (8t + 8) \, dt\). We thus have

$$
\int_{-3}^{2} (2t + 2) \sqrt{4t^2 + 8t + 5} \, dt = \frac{1}{4} \int_{17}^{37} \sqrt{u} \, du \\
= \left( \frac{1}{4} \right) \left( \frac{2}{3} \right) u^{3/2} \bigg|_{17}^{37} \\
= \frac{1}{6} (37^{3/2} - 17^{3/2}) \\
\approx 25.83.
$$

2. Find the work done by the force field

\[
\mathbf{F}(x,y,z) = \mathbf{i} - 2ze^{3t} \mathbf{j} - 2ye^{3t} \mathbf{k}
\]
on an object that moves along a straight line path from the point \((-2, 0, -1)\) to the point \((1, 3, 3)\). Your solution must include

a. a parameterization of the path of motion, \(C\)
b. a line integral that gives the work
c. evaluation of the integral.

**Solution:** The path of motion, \(C\), can be parameterized as

\[
x = -2 + 3t \\
y = 3t \\
z = -1 + 4t \\
0 \leq t \leq 1
\]

and we can also write this parametrization in the vector form

\[
\mathbf{r}(t) = (-2 + 3t) \mathbf{i} + 3t \mathbf{j} + (-1 + 4t) \mathbf{k}.
\]

The tangent vector at each point is given by

\[
\mathbf{r}'(t) = 3 \mathbf{i} + 3 \mathbf{j} + 4 \mathbf{k}
\]

and we see that

\[
|r'(t)| = \sqrt{3^2 + 3^2 + 4^2} = \sqrt{34}.
\]

The unit tangent vector is thus

\[
\mathbf{T}(t) = \frac{1}{\sqrt{34}} (3 \mathbf{i} + 3 \mathbf{j} + 4 \mathbf{k}).
\]

We now note that on the path \(C\) we have

\[
\mathbf{F}(x(t), y(t), z(t)) = \mathbf{i} - 2(-1 + 4t)e^{3t(-1+4t)} \mathbf{j} - 2(3t)e^{3t(-1+4t)} \mathbf{k}.
\]
Thus on the path $C$ we have
\[
\mathbf{F} \cdot \mathbf{T} = \frac{1}{\sqrt{34}} (3 - 6(-1 + 4t)e^{3(1+4t)} - 24te^{3(1+4t)})
\]
\[
= \frac{1}{\sqrt{34}} (3 - 6(-1 + 4t)e^{3(1+4t)} - 24te^{3(1+4t)})
\]
\[
= \frac{1}{\sqrt{34}} (3 - 2(-3 + 24t)e^{(-3t+12t^2)}).
\]

The work done by the given force field is
\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_0^1 \frac{1}{\sqrt{34}} (3 - 2(-3 + 24t)e^{(-3t+12t^2)}) \sqrt{34} \, dt
\]
\[
= \int_0^1 3 \, dt - 2 \int_0^1 (-3 + 24t)e^{(-3t+12t^2)} \, dt
\]
\[
= 3 - 2 \int_0^1 e^u \, du
\]
\[
= 3 - 2(e^9 - 1)
\]
\[
= 5 - 2e^9.
\]

3. For the same force field used in problem 2:
\[
\mathbf{F}(x,y,z) = i - 2ze^{yz}j - 2ye^{yz}k,
\]
\[a.\] Show that $\mathbf{F}$ satisfies the cross partial conditions.
\[b.\] Find a potential function for $\mathbf{F}$.
\[c.\] By appealing to the Fundamental Theorem of Line Integrals, show that the work done by $\mathbf{F}$ in moving an object along any smooth path, $C$, from the point $(-2,0,-1)$ to the point $(1,3,3)$ is the same as the answer that you got in problem 2 (assuming that you did problem 2 correctly).

**Solution:** We note that
\[
\mathbf{F}(x,y,z) = Mi + Nj + Pk
\]
where
\[
M = 1
\]
\[
N = -2ze^{yz}
\]
\[
P = -2ye^{yz}
\]
and we observe that
\[
M_y = 0 = N_z
\]
\[
N_z = -2yze^{yz} - 2e^{yz} = P_y
\]
\[
P_x = 0 = M_z.
\]
Thus the cross partial conditions are satisfied. 
If $f(x,y,z)$ is a potential function for $\mathbf{F}$, then
\[
\nabla f = f_x i + f_y j + f_z k = Mi + Nj + Pk.
\]
We must have $f_x = M$ which means that $f_x = 1$ and hence $f(x,y,z) = x + g(y,z)$ for
some function $g$. This gives $f_y = g_y$ but we also must have $f_y = N$ and thus
$$g_y = -2ze^{yz}.$$ Integration with respect to $y$ gives
$$g(y,z) = -2e^{yz} + h(z).$$
Thus we see that
$$f(x,y,z) = x - 2e^{yz} + h(z)$$
This gives $f_z = -2ye^{yz} + h'(z)$ but we also must have $f_z = -2ye^{yz}$ and thus $h'(z) = 0$ which means that $h(z) = K$ for some constant $K$. We only need one potential function, so we can take $K = 0$. Thus a potential function for $f$ is
$$f(x,y,z) = x - 2e^{yz}.$$ By the Fundamental Theorem of Line Integrals, the work done by $F$ in moving an object along any smooth path from the point $(-2,0,-1)$ to the point $(1,3,3)$ is
$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = f(1,3,3) - f(-2,0,-1)$$
$$= (1 - 2e^9) - (-2 - 2e^0)$$
$$= 5 - 2e^9.$$