Recall the following properties of dot products: If \( \mathbf{u} \) and \( \mathbf{v} \) are any vectors and \( k \) is any scalar, then

1) \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2) \( (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v}) \)
3) \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)
4) \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \)
5) \( \mathbf{u} \cdot \mathbf{0} = 0 \).

Use the above properties (whichever are needed) to prove that if \( \mathbf{u} \) and \( \mathbf{v} \) are any vectors, then

\[
|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2.
\]

You must write the proof step-by-step with a little note at the end of each line of the proof indicating which of the above five properties you are using in that step of the proof. Remember to write “=” where needed!

**Proof:**

\[
|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \text{ by property 4}
\]

\[
= (\mathbf{u} + (\mathbf{-v})) \cdot (\mathbf{u} + (\mathbf{-v})) \text{ by definition of } \mathbf{u} - \mathbf{v}
\]

\[
= (\mathbf{u} + (\mathbf{-v})) \cdot \mathbf{u} + (\mathbf{u} + (\mathbf{-v})) \cdot (\mathbf{-v}) \text{ by property 3}
\]

\[
= \mathbf{u} \cdot (\mathbf{u} + (\mathbf{-v})) + (\mathbf{-v}) \cdot (\mathbf{u} + (\mathbf{-v})) \text{ by property 1}
\]

\[
= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot (\mathbf{-v}) + (\mathbf{-v}) \cdot \mathbf{u} + (\mathbf{-v}) \cdot (\mathbf{-v}) \text{ by property 3}
\]

\[
= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \text{ by property 2}
\]

\[
= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \text{ by property 1}
\]

\[
= |\mathbf{u}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \text{ by property 4.}
\]

This completes the proof.