MATH 2203 – Exam 2 (Version 2) Solutions
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Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. For the motion of a particle whose position at time \( t \) is given by

\[ \mathbf{r}(t) = 2 \cos(t) \mathbf{i} + 3 \sin(t) \mathbf{j} + 4t \mathbf{k}, \]

show how to compute

(a) the velocity vector, \( \mathbf{v}(t) \)
(b) the acceleration vector, \( \mathbf{a}(t) \)
(c) the speed, \( v(t) \)
(d) the unit tangent vector (also called the direction of motion), \( \mathbf{T}(t) \)

Solution: The velocity vector is

\[ \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -2 \sin(t) \mathbf{i} + 3 \cos(t) \mathbf{j} + 4 \mathbf{k}. \]

The acceleration vector is

\[ \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -2 \cos(t) \mathbf{i} - 3 \sin(t) \mathbf{j}. \]

The speed is

\[ v(t) = |\mathbf{v}(t)| = \sqrt{4 \sin^2(t) + 9 \cos^2(t) + 16}. \]

The unit tangent vector is

\[ \mathbf{T}(t) = \frac{1}{v(t)} \mathbf{v}(t) = \frac{1}{\sqrt{4 \sin^2(t) + 9 \cos^2(t) + 16}} (-2 \sin(t) \mathbf{i} + 3 \cos(t) \mathbf{j} + 4 \mathbf{k}). \]

2. What two angles of elevation will enable a projectile to reach a target 16 km downrange on the same level as the gun if the projectile’s initial speed is 400 m/sec? (Take the acceleration due to gravity to be 9.8 m/sec\(^2\).)

(You must include all details needed to arrive at your answer to this question.)
Solution: Based on formulas that we have derived in class several times, the $x$ and $y$ coordinates of the projectile at time $t$ are

\[
x = 400 \cos (\alpha) t \\
y = 400 \sin (\alpha) t - \frac{1}{2} gt^2
\]

where $\alpha$ is the firing angle. Since we want to have $y = 0$ when $x = 16,000$, we must find $t$ such that

\[400 \cos (\alpha) t = 16,000.
\]

Solving this gives

\[t = \frac{40}{\cos (\alpha)}.
\]

By putting this value of $t$ into the above $y$ equation and then setting $y = 0$ we obtain

\[400 \sin (\alpha) \left( \frac{40}{\cos (\alpha)} \right) - \frac{1}{2} g \left( \frac{40}{\cos (\alpha)} \right)^2 = 0.
\]

This gives

\[16,000 \sin (\alpha) \cos (\alpha) = 800g
\]

or

\[10 \cdot 2 \sin (\alpha) \cos (\alpha) = g
\]

or

\[\sin (2\alpha) = \frac{g}{10}.
\]

From this we obtain

\[2\alpha = \arcsin \left( \frac{g}{10} \right) \quad \text{or} \quad 2\alpha = 180^\circ - \arcsin \left( \frac{g}{10} \right).
\]

By using a calculator we see that the two angles in question are

\[\alpha \approx 39.26^\circ \quad \text{and} \quad \alpha \approx 50.74^\circ.
\]

3. Find the length of the portion of the curve

\[\mathbf{r} (t) = t \cos (t) \mathbf{i} + t \sin (t) \mathbf{j} + \frac{2\sqrt{2}}{3} t^{3/2} \mathbf{k}
\]

that corresponds to $0 \leq t \leq \pi$. (All details of computations must be included.)

Solution: First note that

\[\mathbf{v} (t) = (-t \sin (t) + \cos (t)) \mathbf{i} + (t \cos (t) + \sin (t)) \mathbf{j} + \sqrt{2} t^{1/2} \mathbf{k}
\]
and that

\[ v(t) = |v(t)| = \sqrt{(-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2 + 2t} \]

\[ = \sqrt{t^2 \sin^2(t) + t^2 \cos^2(t) + \cos^2(t) + \sin^2(t) + 2t} \]

\[ = \sqrt{t^2 + 2t + 1} \]

\[ = \sqrt{(t + 1)^2} \]

\[ = t + 1. \]

The arc length is thus

\[ \int_0^\pi v(t) \, dt = \int_0^\pi (t + 1) \, dt \]

\[ = \left( \frac{1}{2} t^2 + t \right) \bigg|_{t=0}^{t=\pi} \]

\[ = \frac{1}{2} \pi^2 + \pi. \]

4. Compute the curvature function for the curve

\[ \mathbf{r}(t) = 6 \sin(2t) \mathbf{i} + 6 \cos(2t) \mathbf{j} + 5t \mathbf{k}. \]

(All details must be included.)

**Solution:** For this curve,

\[ \mathbf{v}(t) = 12 \cos(2t) \mathbf{i} - 12 \sin(2t) \mathbf{j} + 5 \mathbf{k} \]

and

\[ v(t) = \sqrt{144 \cos^2(2t) + 144 \sin^2(2t) + 25} = 13 \]

so the unit tangent vector is

\[ \mathbf{T}(t) = \frac{1}{13} (12 \cos(2t) \mathbf{i} - 12 \sin(2t) \mathbf{j} + 5 \mathbf{k}). \]

Furthermore

\[ \frac{d\mathbf{T}}{dt} = \frac{1}{13} (-24 \sin(2t) \mathbf{i} - 24 \cos(2t) \mathbf{j}) \]

and

\[ \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{13} \sqrt{24^2 \sin^2(2t) + 24^2 \cos^2(2t)} = \frac{24}{13}. \]

The curvature function is thus

\[ \kappa(t) = \frac{1}{v(t)} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{24}{169}. \]
5. For the position function

\[ \mathbf{r}(t) = (t + 1) \mathbf{i} + 2t \mathbf{j} + t^2 \mathbf{k}, \]

find the tangential component of acceleration \((a_T(t))\) and the normal component of acceleration \((a_N(t))\). (Show all details of how you do this.)

**Solution:** We have

\[ \mathbf{v}(t) = \mathbf{i} + 2 \mathbf{j} + 2t \mathbf{k} \]

and

\[ v(t) = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2}. \]

The tangential component of acceleration is thus

\[ a_T(t) = \frac{dv}{dt} = \frac{4t}{\sqrt{5 + 4t^2}}. \]

Also

\[ \mathbf{a}(t) = 2 \mathbf{k} \]

and thus \(|\mathbf{a}(t)| = |2 \mathbf{k}| = 2\). Therefore the normal component of acceleration is

\[ a_N(t) = \sqrt{|\mathbf{a}(t)|^2 - a_T(t)^2} \]

\[ = \sqrt{4 - \left( \frac{4t}{\sqrt{5 + 4t^2}} \right)^2} \]

\[ = \sqrt{\frac{20}{5 + 4t^2}}. \]