MATH 2203 – Exam 3 (Version 1) Solutions
March 21, 2011
S. F. Ellermeyer

Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes. (Keep your eyes on your own paper.)

1. In answering this question, please use complete sentences and include any pictures that might be helpful in explaining your answers.

a) What is the largest possible subset of \( \mathbb{R}^2 \) that can be used as a domain in defining a function by
\[
f(x, y) = \sqrt{x^2 + y^2}.
\]

b) What is the range of \( f \)?

c) Describe the level curves of this function. Include a picture that shows the general nature of the level curves and also include a written description of the form “The level curves of this function are _____”.

Answers: Since \( x^2 + y^2 \geq 0 \) no matter what we choose for \((x, y)\), then there is no danger that we will be taking the square root of a negative number. This means that we can use \( \mathbb{R}^2 \) itself as a domain for \( f \). Also, we see that the range of \( f \) is \([0, \infty)\) because if \( c \) is any number in \([0, \infty)\), then the equation \( \sqrt{x^2 + y^2} = c \) can be written as \( x^2 + y^2 = c^2 \) and this equation obviously has many solutions. In fact the solutions are all points lying on the circle of radius \( c \) centered at \((0, 0)\). This also answers part c. The level curves of \( f \) are circles centered at the origin in \( \mathbb{R}^2 \).

2. By considering different paths of approach, show that
\[
\lim_{(x,y) \to (0,0)} \frac{xy}{|xy|}
\]
does not exist. Don’t just show calculations. Explain what you are doing.

Solution: First let us compute this limit along the path \( y = x \). We obtain
\[
\lim_{(x,y) \to (0,0)}_{\text{along } y=x} \frac{x^2}{|x^2|} = \lim_{x \to 0} \frac{x^2}{|x^2|} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1.
\]
Next we compute the limit along the path $y = -x$:

$$
\lim_{(x,y) \to (0,0) \text{ along } y = -x} \frac{-x^2}{-x^2} = \lim_{x \to 0} \frac{-x^2}{x^2} = \lim_{x \to 0} (-1) = -1.
$$

Since we obtained two different limits by using two different paths of approach, we have shown that the limit in question does not exist.

3. For the function

$$
 f(x, y) = \left(x^3 + \frac{y^2}{2}\right)^{2/3},
$$

find the partial derivatives $f_x$ and $f_y$. After you have done this, verify that $f_x(4, 0) = 8$ and $f_y(4, 0) = 1/12$.

**Solution:**

$$
 f_x = \frac{2}{3} \left(x^3 + \frac{y^2}{2}\right)^{-1/3} (3x^2) = 2x^2 \left(x^3 + \frac{y^2}{2}\right)^{-1/3}
$$

and

$$
 f_y = \frac{2}{3} \left(x^3 + \frac{y^2}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3} \left(x^3 + \frac{y^2}{2}\right)^{-1/3}.
$$

Thus

$$
 f_x(4, 0) = 2 (4)^2 \left(4^3 + \frac{0}{2}\right)^{-1/3} = 8
$$

and

$$
 f_y(4, 0) = \frac{1}{3} \left(4^3 + \frac{0}{2}\right)^{-1/3} = \frac{1}{12}.
$$

4. Suppose that

$$
 w = u^2 + v^3 \\
 u = x^3 + y^4 \\
 v = x^4 + y^5.
$$

Find $\partial w / \partial x$. After you have done this, verify that

$$
 \frac{\partial w}{\partial x} \bigg|_{(x,y)=(1,-1)} = 12.
$$

**Solution:**

$$
 \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} \\
 = (2u) (3x^2) + (3v^2) (4x^3) \\
 = 2 \left(x^3 + y^4\right) (3x^2) + 3 \left(x^4 + y^5\right)^2 (4x^3).
$$

Thus

$$
 \frac{\partial w}{\partial x} \bigg|_{(x,y)=(1,-1)} = 2 \left((1)^3 + (-1)^4\right) (3(1)^2) + 3 \left((1)^4 + (-1)^5\right)^2 (4(1)^3) = 12.
$$
5. Let \( \mathbf{u} \) be the unit vector \( \mathbf{u} = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \).

Let \( f \) be the function

\[
f(x, y) = 2x^2 + y^2.
\]

Let \( P \) be the point \((-1, 1)\).

(a) Find \( \nabla f(P) \).

(b) Find \( D_u f(P) \).

(c) In what direction does \( f \) have its greatest (most positive) rate of change at \( P \)? Find the rate of change of \( f \) at \( P \) in this direction.

(d) In what two directions does \( f \) have a zero rate of change at \( P \)?

Answers:

\[
\nabla f = f_x \mathbf{i} + f_y \mathbf{j} = 4x \mathbf{i} + 2y \mathbf{j}
\]

and thus

\[
\nabla f(P) = -4 \mathbf{i} + 2 \mathbf{j}.
\]

Therefore

\[
D_u f(P) = \nabla f(P) \cdot \mathbf{u} = (-4) \left( \frac{3}{5} \right) + (2) \left( \frac{4}{5} \right) = -\frac{4}{5}.
\]

\( f \) has its greatest rate of change in the direction of \( \nabla f(P) \). The rate of change of \( f \) in this direction is

\[
|\nabla f(P)| = \sqrt{(-4)^2 + (2^2)} = \sqrt{20}.
\]

The directions of zero rate of change are the two directions that are orthogonal to \( \nabla f(P) \). These are the directions of \( 2 \mathbf{i} + 4 \mathbf{j} \) and \(-2 \mathbf{i} - 4 \mathbf{j} \).