Path Independence, Potential Functions, and Conservative Vector Fields

Outline of Section 13.3 of Hass, Weir, Thomas
Definition

Let $\mathbf{F}$ be a vector field defined on an open set $D$ in $\mathbb{R}^3$ (or $\mathbb{R}^2$). We say that $\mathbf{F}$ is a conservative vector field and we also say that $\int \mathbf{F} \cdot d\mathbf{r}$ is path–independent if for any two points, $A$ and $B$, in $D$, the value of

$$\int_{C} \mathbf{F} \cdot d\mathbf{r}$$

is the same for all choices smooth curves, $C$, connecting $A$ to $B$. 
Example

Let $D = R^2$ and let $\mathbf{F}(x, y) = -10\mathbf{j}$.
Show that $\mathbf{F}$ is conservative and that
$\int \mathbf{F} \cdot d\mathbf{r}$ is path–independent.
Definition

If $f$ is a real–valued function defined on $D$ and
$F(x, y, z) = \nabla f(x, y, z)$ for all points $(x, y, z) \in D$,
then we say that $f$ is a potential function for $F$ on $D$. 
Example

Let $D = \mathbb{R}^2$ and let $F(x,y) = -10\mathbf{j}$.
Find a potential function for $F$ on $D$. 
Example

Let $D = \mathbb{R}^3$ and let

$$F(x, y, z) = (e^x \cos(y) + yz)i + (xz - e^x \sin(y))j + (xy + z)k.$$ 

Find a potential function for $F$ on $D$. 

Theorem (The Fundamental Theorem of Calculus for Line Integrals)

Let \( C \), given by \( \mathbf{r}(t) \), be a smooth curve in space or in the plane jointing the point \( A \) to the point \( B \). Let \( f \) be a differentiable function with a continuous gradient vector, \( \mathbf{F} = \nabla f \) on a domain containing the curve \( C \). Then

\[
\int_{C} \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).
\]
Example

Let $D = \mathbb{R}^2$ and let $F(x, y) = -10\mathbf{j}$.

Let $C$ be any smooth curve in $\mathbb{R}^2$ joining the point $(0, 5)$ to the point $(5, 0)$.

Compute

$$\int_C F \cdot \, dr.$$
Example

Let $D = \mathbb{R}^3$ and let

$$\mathbf{F}(x, y, z) = (e^x \cos(y) + yz)\mathbf{i} + (xz - e^x \sin(y))\mathbf{j} + (xy + z)\mathbf{k}.$$ 

Let $C$ be any smooth curve in $\mathbb{R}^3$ joining the point $(-1, 3, 9)$ to the point $(1, 6, -4)$.

Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$
Theorem

Let \( \mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k} \) be a vector field on a connected and simply connected domain and suppose that the component functions \((M,N, P)\) all have continuous first order partial derivatives on \( D \). Then the following statements are equivalent (meaning that these statements are either all true or all false).

1) \( P_y = N_z \) and \( M_z = P_x \) and \( N_x = M_y \) at all points \( (x,y,z) \in D \).
2) There exists a function \( f \) on \( D \) such that \( \mathbf{F} = \nabla f \).
3) \( \mathbf{F} \) is conservative on \( D \).
4) \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \) around every closed loop, \( C \), that lies in \( D \).
Remark

If $F$ is conservative on $D$, and $A$ and $B$ are any two points in $D$, and $C$ is any smooth curve that joins $A$ to $B$ and is contained in $D$, then it makes sense to write

$$\int_C F \cdot dr = \int_A^B F \cdot dr$$

because the value of the integral does not depend on the curve $C$ that is chosen (as long as the curve joins $A$ to $B$). This integral can also be written as

$$\int_A^B Mdx + Ndy + Pdz.$$
Definition

Any expression of the form

\[ M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz \]

is called a **differential form**. A differential form is said to be **exact** on a domain \( D \) if

\[ Mdx + Ndy + Pdz = f_x dx + f_y dy + f_z dz \]

for some function \( f \) at all points \( (x,y,z) \in D \).
Example

Show that the differential form

\[ yz \, dx + xz \, dy + xy \, dz \]

is exact and then evaluate the integral

\[ \int_{(1,1,2)}^{(3,5,0)} yz \, dx + xz \, dy + xy \, dz. \]