Some Basic Properties of Vectors in \( \mathbb{R}^n \)

The following ten facts about \( \mathbb{R}^n \) are (rather obviously) true:

1. If \( u \) and \( v \) are any two vectors in \( \mathbb{R}^n \), then the vector \( u + v \) is also in \( \mathbb{R}^n \). (In other words, \( \mathbb{R}^n \) is \textit{closed under vector addition}.)
2. If \( u \) is any vector in \( \mathbb{R}^n \) and \( c \) is any scalar, then the vector \( cu \) is also in \( \mathbb{R}^n \). (In other words, \( \mathbb{R}^n \) is \textit{closed under scalar multiplication}.)
3. If \( u \) and \( v \) are any two vectors in \( \mathbb{R}^n \), then \( u + v = v + u \). (In other words, vector addition in \( \mathbb{R}^n \) is \textit{commutative}.)
4. If \( u, v, \) and \( w \) are any three vectors in \( \mathbb{R}^n \), then \( (u + v) + w = u + (v + w) \). (In other words, vector addition in \( \mathbb{R}^n \) is \textit{associative}.)
5. There is a vector in \( \mathbb{R}^n \) called the zero vector (and denoted by \( 0 \)) such that if \( u \) is any vector in \( \mathbb{R}^n \), then \( u + 0 = u \). (In other words, a \textit{zero vector exists} in \( \mathbb{R}^n \).)
6. For each vector \( u \) in \( \mathbb{R}^n \), there is a vector called the additive inverse of \( u \) (and denoted by \( -u \)) such that \( u + (-u) = 0 \). (In other words, each vector in \( \mathbb{R}^n \) has an \textit{additive inverse}.)
7. If \( u \) and \( v \) are any two vectors in \( \mathbb{R}^n \) and \( c \) is any scalar, then \( c(u + v) = cu + cv \). (In other words, scalar multiplication is \textit{distributive} over vector addition in \( \mathbb{R}^n \).)
8. If \( u \) is any vector in \( \mathbb{R}^n \) and \( c \) and \( d \) are any two scalars, then \( (c + d)u = cu + du \).
9. If \( u \) is any vector in \( \mathbb{R}^n \) and \( c \) and \( d \) are any two scalars, then \( (cd)u = c(du) \).
10. If \( u \) is any vector in \( \mathbb{R}^n \), then \( 1u = u \).

Any collection of objects (not just \( \mathbb{R}^n \)) that satisfies all ten of the above requirements is called a \textit{vector space}. Some of the most interesting (and important in higher applications) vector spaces are \textit{function spaces}. Function Spaces are sets of \textit{functions} (that satisfy all ten of the above properties). Just as with \( \mathbb{R}^n \), in any vector space we have such notions as linear combinations and linear independence. We also have the notion of linear transformations.

We will look at several examples of vector spaces (other than \( \mathbb{R}^n \)). Some of these examples will be subspaces of \( \mathbb{R}^n \) and some will be subspaces of other vector spaces. A \textit{subspace} of a vector space is a subset of the vector space which is itself a vector space. To begin, we give definitions of the terms “vector space” and “subspace”.
Definitions of “Vector Space” and “Subspace”

A vector space, $V$, over the field of real numbers (called scalars) is a set of objects (called vectors) equipped with operations of vector addition and scalar multiplication such that all ten of the following properties are satisfied:

1. If $u$ and $v$ are any two vectors in $V$, then the vector $u + v$ is also in $V$. (In other words, $V$ is closed under vector addition.)
2. If $u$ is any vector in $V$ and $c$ is any scalar, then the vector $cu$ is also in $V$. (In other words, $V$ is closed under scalar multiplication.)
3. If $u$ and $v$ are any two vectors in $V$, then $u + v = v + u$. (In other words, vector addition in $V$ is commutative.)
4. If $u$, $v$, and $w$ are any three vectors in $V$, then $(u + v) + w = u + (v + w)$. (In other words, vector addition in $V$ is associative.)
5. There is a vector in $V$ called the zero vector (and denoted by $0$) such that if $u$ is any vector in $V$, then $u + 0 = u$. (In other words, a zero vector exists in $V$.)
6. For each vector $u$ in $V$, there is a vector called the additive inverse of of $u$ (and denoted by $-u$) in $V$ such that $u + (-u) = 0$. (In other words, each vector in $V$ has an additive inverse.)
7. If $u$ and $v$ are any two vector in $V$ and $c$ is any scalar, then $c(u + v) = cu + c v$. (In other words, scalar multiplication is distributive over vector addition in $V$.)
8. If $u$ is any vector in $V$ and $c$ and $d$ are any two scalars, then $(c + d)u = cu + du$.
9. If $u$ is any vector in $V$ and $c$ and $d$ are any two scalars, then $(cd)u = c(du)$.
10. If $u$ is any vector in $V$, then $1u = u$.

If $V$ is a vector space and $W$ is a vector space that is a subset of $V$, then $W$ is called a subspace of $V$. 


Example: A line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

Any line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. As a specific example, let's suppose that $v \in \mathbb{R}^2$ is the vector

$$v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and then define $W$ to be the set of all scalar multiples of $v$. Then $W$ is a line through the origin in $\mathbb{R}^2$. To see that $W$ is a subspace of $\mathbb{R}^2$, let us check that all ten of the requirements for $W$ to be a vector space are satisfied:

1. If we take any two vectors in $W$ and add them together, then we get another vector in $W$. Thus, $W$ is closed under vector addition.
2. If we take any vector in $W$ and multiply it by a scalar, then we get another vector in $W$. Thus, $W$ is closed under scalar multiplication.
3. Since vector addition in $\mathbb{R}^2$ is commutative and since $W \subseteq \mathbb{R}^2$, then vector addition in $W$ is commutative.
4. Since vector addition in $\mathbb{R}^2$ is associative and since $W \subseteq \mathbb{R}^2$, then vector addition in $W$ is associative.
5. The zero vector in $\mathbb{R}^2$ is

$$0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

This vector is also in $W$ because

$$0 = 0 \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$ 

Thus, $W$ has a zero vector.
6. If $u$ is any vector in $W$, then $u = tv$ for some scalar $t$. Note that $W$ also contains the vector $-tv$ and that this vector is the additive inverse of $u$. Thus every vector in $W$ has an additive inverse in $W$.
7. Since the distributive property holds for all vectors in $\mathbb{R}^2$, it also holds for all vectors in $W$.
8. Since the property $(c + d)u = cu + du$ holds for all vectors $u \in \mathbb{R}^2$, then it also holds for all vectors $u \in W$.
9. Since the property $(cd)u = c(du)$ holds for all vectors $u \in \mathbb{R}^2$, then it also holds for all vectors $u \in W$.
10. Since the property $1u = u$ holds for all vectors $u \in \mathbb{R}^2$, then it also holds for all vectors $u \in W$.

We have verified that $W$ is a vector space. Thus, $W$ is a subspace of $\mathbb{R}^2$.

Notice that, in the above example, the only properties that required a little bit of
work to prove were properties 1) closure under vector addition, 2) closure under scalar multiplication, 5) existence of a zero vector, and 6) existence of additive inverses. The rest of the properties were simply “inherited” from the vector space $\mathbb{R}^2$. In fact, we will see that it is really only necessary to verify properties 1 and 2. Specifically, we will see that if we start with a known vector space, $V$, and we want to prove that a certain non–empty subset, $W$, of $V$ is a subspace of $V$, then it is only necessary to prove that $W$ is closed under vector addition and scalar multiplication. Before proving this fact, let us look at several more examples of vector spaces and some of their subspaces. In the meanwhile, let us accept it as true that if $V$ is a vector space and $W$ is a non–empty subset of $V$ that is closed under vector addition and scalar multiplication, then $W$ is a subspace of $V$. 
Example: The first quadrant in $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^2$.

The first quadrant in $\mathbb{R}^2$ consists of all vectors

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

such that $x > 0$ and $y > 0$. Let us give the name $W$ to the first quadrant. It is obvious that $W$ is a subset of $\mathbb{R}^2$ because every vector in $W$ is also in $\mathbb{R}^2$. However, $W$ is not a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication. For example, the vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \in W$$

but the vector

$$-2\mathbf{u} = \begin{bmatrix} -2 \\ -12 \end{bmatrix} \notin W.$$
Example: The nullspace of a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a subspace of \( \mathbb{R}^n \).

The nullspace, \( nul(T) \), of a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is defined to be the set of all vectors in \( \mathbb{R}^n \) that are mapped onto the zero vector (in \( \mathbb{R}^m \)) by \( T \). More formally,

\[
nul(T) = \{ x \in \mathbb{R}^n \mid T(x) = 0_m \}. \]

To prove that \( nul(T) \) is a subspace of \( \mathbb{R}^n \), we need to prove that \( nul(T) \) is closed under vector addition and scalar multiplication:

1. If \( x \) and \( y \) are any two vectors in \( nul(T) \), then
   \[
   T(x + y) = T(x) + T(y) = 0_m + 0_m = 0_m,
   \]
   which shows that \( x + y \) is in \( nul(T) \). Thus, \( nul(T) \) is closed under vector addition.

2. If \( x \) is any vector in \( nul(T) \) and \( c \) is any scalar, then
   \[
   T(cx) = cT(x) = c \cdot 0_m = 0_m,
   \]
   which shows that \( cx \) is in \( nul(T) \). Thus \( nul(T) \) is closed under scalar multiplication.

Example: The range of a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a subspace of \( \mathbb{R}^m \).

The range, \( \text{ran}(T) \), of a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) is defined to be the set of all vectors in \( \mathbb{R}^m \) that are the images under \( T \) of at least one vector in \( \mathbb{R}^n \). More formally,

\[
\text{ran}(T) = \{ b \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \mid b = T(x) \}.
\]

To prove that \( \text{ran}(T) \) is a subspace of \( \mathbb{R}^m \), we need to prove that \( \text{ran}(T) \) is closed under vector addition and scalar multiplication:

1. If \( a \) and \( b \) are any two vectors in \( \text{ran}(T) \), then there are vectors \( x \) and \( y \) in \( \mathbb{R}^n \) such that
   \[
a = T(x) \quad \text{and} \quad b = T(y).
   \]
   This means that
   \[
a + b = T(x) + T(y) = T(x + y),
   \]
   and since \( x + y \) is in \( \mathbb{R}^n \), then \( a + b \) is in \( \text{ran}(T) \). Thus, \( \text{ran}(T) \) is closed under vector addition.

2. If \( a \) is any vector in \( \text{ran}(T) \) and \( c \) is any scalar, then there is a vector \( x \in \mathbb{R}^n \) such that
   \[
a = T(x).
   \]
   Thus,
   \[
   ca = cT(x) = T(cx),
   \]
   and since \( cx \in \mathbb{R}^n \), we can conclude that \( ca \in \text{ran}(T) \). Thus \( \text{ran}(T) \) is closed.
under scalar multiplication.
Example: The span of any non-empty, finite set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in $\mathbb{R}^n$. Recall that $\text{Span}(S)$ is the set of all linear combinations of vectors in $S$. We will prove that $\text{Span}(S)$ is a subspace of $\mathbb{R}^n$.

1. Let $u$ and $w$ be two vectors in $\text{Span}(S)$. Then $u$ can be written as

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

where $c_1, c_2, \ldots, c_k$ are scalars, and $w$ can be written as

$$w = d_1v_1 + d_2v_2 + \cdots + d_kv_k$$

where $d_1, d_2, \ldots, d_k$ are scalars.

We now observe that

$$u + w = (c_1v_1 + c_2v_2 + \cdots + c_kv_k) + (d_1v_1 + d_2v_2 + \cdots + d_kv_k)$$

$$= (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k$$

which means that $u + w$ is a linear combination of the vectors in $S$. In other words, $u + w$ is in $\text{Span}(S)$. This shows that $\text{Span}(S)$ is closed under vector addition.

2. Let $u$ be a vector in $\text{Span}(S)$ and let $c$ be a scalar. Since $u$ is in $\text{Span}(S)$, we know that $u$ can be written as

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

where $c_1, c_2, \ldots, c_k$ are scalars.

This implies that

$$cu = c(c_1v_1 + c_2v_2 + \cdots + c_kv_k)$$

$$= (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_k)v_k,$$

showing that $cu$ is a linear combination of the vectors in $S$, and hence that $cu$ is in $\text{Span}(S)$. Therefore, $\text{Span}(S)$ is closed under scalar multiplication.
**Example:** The set of all $m \times n$ matrices is a vector space.

The set, $M_{m\times n}$, of all $m \times n$ matrices is a vector space with the usual operations of matrix addition and multiplication of matrices by scalars.

Without going through all of the details of verifying that all ten vector space axioms are satisfied, let us just make a few observations: For example, it is clear that $M_{m\times n}$ is closed under vector addition, because if we add two $m \times n$ matrices, then we get another $m \times n$ matrix. It is also clear that $M_{m\times n}$ is closed under scalar multiplication, because if we multiply an $m \times n$ matrix by a scalar, then we get another $m \times n$ matrix. Also, it is clear that the $m \times n$ matrix with all entries equal to zero serves as the zero vector in the vector space $M_{m\times n}$. As a final observation, the additive inverse of an $m \times n$ matrix, $A$, is the matrix whose entries are the negatives of the entries of $A$.

In fact, the vector space $M_{m\times n}$ is very similar to the vector space $\mathbb{R}^p$ where $p = m \times n$. 
Function Spaces

A function space is a vector space whose “vectors” are actually functions.

Vector addition in a function space is defined as follows: If $f$ and $g$ are functions that have the same domain, $D$, then $f + g$ is the function with domain $D$ defined by

$$(f + g)(x) = f(x) + g(x)$$ for all $x \in D$.

Scalar multiplication in a function space is defined as follows: If $f$ is a function with domain $D$, and $c$ is a scalar, then $cf$ is the function with domain $D$ defined by

$$(cf)(x) = cf(x)$$ for all $x \in D$.

Example: The set of all functions having a common domain is a vector space.

Let $X$ be any subset of the real numbers and let $F(X)$ be the set of all functions with domain $X$ and codomain $\mathbb{R}$. In other words, let $F(X)$ be the set of all functions, $f$, such that $f : X \to \mathbb{R}$. Then $F(X)$ is a vector space.

1. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, then $f + g : X \to \mathbb{R}$. This shows that $F(X)$ is closed under vector addition.
2. If $f : X \to \mathbb{R}$ and $c$ is a scalar, then $cf : X \to \mathbb{R}$. This shows that $F(X)$ is closed under scalar multiplication.

We remark that the zero vector in $F(X)$ is the function $z : X \to \mathbb{R}$ defined by $z(x) = 0$ for all $x \in X$. Also, for any $f \in F(X)$, the additive inverse of $f$ is the function $-f \in F(X)$ defined by $(-f)(x) = -1 \cdot f(x)$ for all $x \in X$.

Here is a specific example to illustrate this concept: Suppose that $X$ is the set $[0, \infty)$. In this case, $F(X)$ is the set of all functions, $f$, such that $f : [0, \infty) \to \mathbb{R}$. An example of a function that is in $F(X)$ is

$$f(x) = \sqrt{x}$$ because this function is defined for all $x \in [0, \infty)$.

An example of a function that is not in $F(X)$ is

$$f(x) = \frac{1}{x - 9}$$ because this function is not defined for all $x \in [0, \infty)$.

The zero vector in $F(X)$ is the function defined by $z(x) = 0$ for all $x \in [0, \infty)$.

If $f$ is the function in $F(X)$ defined by $f(x) = e^x \cos(x) - 4x^3$ for all $x \in [0, \infty)$, then the additive inverse of $f$ is the function, $-f$, in $F(X)$ defined by $(-f)(x) = -e^x \cos(x) + 4x^3$ for all $x \in [0, \infty)$. 

Example: The set of all continuous functions having a common domain is a vector space.

Let $X$ be any subset of the real numbers and let $C^0(X)$ be the set of all functions with domain $X$ and codomain $\mathbb{R}$ that are continuous on $X$. (To say that a function, $f$, is “continuous on $X$” means that $f$ is continuous at all points of the set $X$.)

$C^0(X)$ is a vector space.

1. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ and $f$ and $g$ are both continuous on $X$, then $f + g : X \to \mathbb{R}$ and $f + g$ is also continuous on $X$. This shows that $C^0(X)$ is closed under vector addition. (The fact that the sum of two continuous functions is continuous is a theorem from Calculus.)

2. If $f : X \to \mathbb{R}$ is continuous on $X$, and $c$ is a scalar, then $cf : X \to \mathbb{R}$ and $cf$ is also continuous on $X$. This shows that $C^0(X)$ is closed under scalar multiplication. (The fact that a scalar multiple of a continuous function is a continuous function is a theorem from Calculus.)

Note that $C^0(X)$ is a subspace of $F(X)$. 


Example: The set of all continuously differentiable functions having a common domain is a vector space.

Let \( X \) be any subset of the real numbers and let \( C^1(X) \) be the set of all functions with domain \( X \) and codomain \( \mathbb{R} \) that are continuously differentiable on \( X \). (To say that \( f \) is “continuously differentiable on \( X \)” means that \( f \) is differentiable at all points of the set \( X \) and that \( f' \) is a continuous function on \( X \).)

\( C^1(X) \) is a vector space.

1. If \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) and \( f \) and \( g \) are both differentiable on \( X \), then \( f + g : X \to \mathbb{R} \) and \( f + g \) is also differentiable on \( X \). This shows that \( C^1(X) \) is closed under vector addition. (The fact that the sum of two differentiable functions is a differentiable function is a theorem from Calculus.)

2. If \( f : X \to \mathbb{R} \) is differentiable on \( X \), and \( c \) is a scalar, then \( cf : X \to \mathbb{R} \) and \( cf \) is also differentiable on \( X \). This shows that \( C^1(X) \) is closed under scalar multiplication. (The fact that a scalar multiple of a differentiable function is a differentiable function is a theorem from Calculus.)

Note that \( C^1(X) \) is a subspace of \( C^0(X) \). The fact that \( C^1(X) \) is a subset of \( C^0(X) \) follows from a Calculus theorem that says that all differentiable functions (on a given set \( X \)) are also continuous (on \( X \)).
Example: The set $P$ of all polynomial functions with a common domain is a vector space.

Let $X$ be a given subset of $\mathbb{R}$.

Recall that a polynomial function, $p$, with domain $X$ is a function defined by a formula of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

for all $x \in X$ where $a_0, a_1, a_2, \ldots, a_n$ are scalars that are called the coefficients of the polynomial function. The integer $n$ is called the degree of the polynomial function. Constant polynomial functions of the form

$$p(x) = a_0$$

for all $x \in X$ are considered to be polynomial functions of degree zero.

The set, $P(X)$, of all polynomial functions $p : X \to \mathbb{R}$ is a vector space. It is in fact a subspace of $C^1(X)$, because all polynomial functions on $X$ are also differentiable on $X$. 
Example: The set of all polynomial functions that have a common domain and that have degree less than or equal to \( n \) is a vector space.

Let \( n \) be a specified non–negative integer.

The set, \( P_n(X) \), of all polynomial functions with domain \( X \) and degree less than or equal to \( n \) is a subspace of \( P(X) \).

To see why \( P_n(X) \) is a vector space, note that if we add two polynomial functions with domain \( X \) and with degree less than or equal to \( n \), then we get another polynomial function with domain \( X \) and with degree less than or equal to \( n \). This explains why \( P_n(X) \) is closed under vector addition. Likewise, \( P_n(X) \) is also closed under scalar multiplication.
An Example of a Set of Functions that is not a Vector Space

An example of a set of functions that is not a vector space is the set of all polynomial functions that have integer coefficients.

For example, the polynomial function

\[ p(x) = 9 - 2x + 3x^2 + 8x^7 \]

is in this set, but the polynomial function \(0.5p\) which is defined by the formula

\[ (0.5p)(x) = 4.5 - x + 1.5x^2 + 4x^7 \]

is not in this set. Thus, this set is not closed under scalar multiplication.
The Subspace Spanned by a Set of Vectors, $S$, in a Vector Space

If $V$ is a vector space and $S = \{v_1, v_2, \ldots, v_k\}$ is a non-empty set of vectors in $V$, then the span of $S$, denoted by $\text{Span}(S)$, is the set of all linear combinations of vectors in $S$. The set $\text{Span}(S)$ is a subspace of $V$.

We have already seen that this is true when $V = \mathbb{R}^n$. The next example shows that the same idea applies, for example in a function space.

Example: Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is the function $f(x) = \sin(x)$ and suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is the function $g(x) = \cos(x)$ and let $S = \{f, g\}$. Then $\text{Span}(S)$ is a subspace of $C^1$.

First note that $\text{Span}(S)$ consists of all function $h : \mathbb{R} \rightarrow \mathbb{R}$ that are linear combinations of the functions $f$ and $g$. Thus, every function $h$ in $\text{Span}(S)$ is defined by a formula of the form

$$h(x) = c_1 \sin(x) + c_2 \cos(x) \text{ for all } x \in \mathbb{R}.$$ 

It is easy to see that if we take two functions that have the above form and add them together, then we get another function of this form. Also, if we multiply a function of this form by a scalar, then we also get another function of this form. Thus, $\text{Span}(S)$ is closed under vector addition and under scalar multiplication.