The Periodically–Forced Harmonic Oscillator

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Abstract
We study the differential equation
\[
\frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy = A \cos (\omega t - \theta)
\]
which models a periodically–forced harmonic oscillator.

1 General Solution of the Unforced Harmonic Oscillator Equation

The harmonic oscillator without external forcing is modelled by the differential equation
\[
\frac{d^2y}{dt^2} + p \frac{dy}{dt} + qy = 0
\]
where \( p = b/m, q = k/m, m \) is the mass of the bob, \( k \) is the spring constant, and \( b \) is the damping coefficient. Due to the physical interpretations of \( p \) and \( q \), we assume that \( p \geq 0 \) and that \( q > 0 \).

The characteristic equation for the differential equation (1) is
\[
\lambda^2 + p\lambda + q = 0
\]
and the eigenvalues are
\[
\lambda_1 = \frac{-p - \sqrt{p^2 - 4q}}{2}
\]
\[
\lambda_2 = \frac{-p + \sqrt{p^2 - 4q}}{2}
\]
Due to the assumptions $p \geq 0$ and $q > 0$, no eigenvalue can have a positive real part. There are thus four possibilities:

1. $\lambda_1 < \lambda_2 < 0$ – in which case the harmonic oscillator is **overdamped** and the general solution of (1) is

   \[ N(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \]

2. $\lambda = \alpha + \beta i$ is an imaginary eigenvalue with $\alpha < 0$ and $\beta > 0$ – in which case the harmonic oscillator is **underdamped** and the general solution of (1) is

   \[ N(t) = c_1 e^{\alpha t} \cos (\beta t) + c_2 e^{\alpha t} \sin (\beta t). \]

3. $\lambda = \beta i$ is an imaginary eigenvalue with $\beta > 0$ – in which case the harmonic oscillator is **undamped** and the general solution of (1) is

   \[ N(t) = c_1 \cos (\beta t) + c_2 \sin (\beta t). \]

4. $\lambda < 0$ is a repeated eigenvalue – in which case the harmonic oscillator is **critically damped** and the general solution of (1) is

   \[ N(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}. \]

Note: We use the notation $N(t)$ to denote the general solution of (1) because we are thinking of this solution as the “natural” response when the harmonic oscillator experiences no external forcing.

## 2 General Solution of the Forced Harmonic Oscillator Equation

If we add an external forcing function, $f(t)$, to the harmonic oscillator, we obtain the differential equation

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = f(t). \] (2)

The general solution of the differential equation (2) is

\[ y(t) = N(t) + F(t) \]
where $N$ is the natural response (the general solution of the corresponding unforced equation) and $F$ is any particular solution of the forced equation (2). We use the notation $F(t)$ to remind us that this a “forced” response. Once we have found a particular forced response, $F(t)$, we will have found the general solution to (2).

3 Some Useful Facts From Trigonometry

**Lemma 1** If $a$ and $b$ are any real numbers with $a \neq 0$, then

$$
\cos \left( \arctan \left( \frac{b}{a} \right) \right) = \frac{|a|}{\sqrt{a^2 + b^2}}
$$

and

$$
\sin \left( \arctan \left( \frac{b}{a} \right) \right) = \frac{b|a|}{a\sqrt{a^2 + b^2}}.
$$

**Proof.** We use the fact that if $x$ is any real number, then

$$
\cos \left( \arctan (x) \right) = \frac{1}{\sqrt{1 + x^2}}
$$

and

$$
\sin \left( \arctan (x) \right) = \frac{x}{\sqrt{1 + x^2}}.
$$

Setting $x = b/a$, we obtain

$$
\cos \left( \arctan \left( \frac{b}{a} \right) \right) = \frac{1}{\sqrt{1 + \left( \frac{b}{a} \right)^2}} = \frac{\sqrt{a^2}}{\sqrt{a^2 + b^2}} = \frac{|a|}{\sqrt{a^2 + b^2}}.
$$

and

$$
\sin \left( \arctan \left( \frac{b}{a} \right) \right) = \frac{\frac{b}{a}}{\sqrt{1 + \left( \frac{b}{a} \right)^2}} = \frac{b|a|}{a\sqrt{a^2 + b^2}}.
$$

**Lemma 2** If $a$, $b$, and $\omega$ are any real numbers, then for all real numbers, $t$, we have

$$
a \cos (\omega t) + b \sin (\omega t) = K \cos (\omega t - \phi)
$$
where

\[ K = \begin{cases} \frac{|a|\sqrt{a^2+b^2}}{a} & \text{if } a \neq 0 \\ \frac{b}{a} & \text{if } a = 0 \end{cases} \]

and

\[ \phi = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a \neq 0 \\ \frac{\pi}{2} & \text{if } a = 0 \end{cases}. \]

**Proof.** First suppose that \( a \neq 0 \) and let

\[ K = \frac{|a|\sqrt{a^2+b^2}}{a} \]

and

\[ \phi = \arctan\left(\frac{b}{a}\right). \]

Then

\[ K \cos (\phi) = K \cos \left(\arctan\left(\frac{b}{a}\right)\right) \]

\[ = \frac{|a|\sqrt{a^2+b^2}}{a} \cdot \frac{|a|}{\sqrt{a^2+b^2}} \]

\[ = a \]

and

\[ K \sin (\phi) = K \sin \left(\arctan\left(\frac{b}{a}\right)\right) \]

\[ = \frac{|a|\sqrt{a^2+b^2}}{a} \cdot \frac{b|a|}{a\sqrt{a^2+b^2}} \]

\[ = b \]

We conclude that

\[ K \cos (\omega t - \phi) = K \left(\cos (\omega t) \cos (\phi) + \sin (\omega t) \sin (\phi)\right) \]

\[ = K \cos (\phi) \cos (\omega t) + K \sin (\phi) \sin (\omega t) \]

\[ = a \cos (\omega t) + b \sin (\omega t) \]

which is what we wanted to prove.
Now we consider the case \( a = 0 \). In this case, we define

\[
K = b
\]

and

\[
\phi = \frac{\pi}{2}
\]

and observe that

\[
K \cos (\phi) = 0 \\
K \sin (\phi) = b.
\]

Thus

\[
K \cos (\omega t - \phi) = K (\cos (\omega t) \cos (\phi) + \sin (\omega t) \sin (\phi)) \\
= K \cos (\phi) \cos (\omega t) + K \sin (\phi) \sin (\omega t) \\
= 0 \cos (\omega t) + b \sin (\omega t) \\
= a \cos (\omega t) + b \sin (\omega t).
\]

\[\square\]

**Remark 3** It is useful to write a function \( a \cos (\omega t) + b \sin (\omega t) \) in the form \( K \cos (\omega t - \phi) \) because it allows us to tell the amplitude and the phase shift of the oscillations very easily. In fact, since

\[
K \cos (\omega t - \phi) = K \cos \left( \omega \left( t - \frac{\phi}{\omega} \right) \right),
\]

we see that the amplitude is \( |K| \) and the phase shift is \( \phi/\omega \). (Also, the period is \( 2\pi/\omega \) and the frequency is \( \omega/(2\pi) \).)

**Example 4** A graph of the function

\[
F(t) = 2 \cos (2t) - 5 \sin (2t)
\]

is shown below.
This function has period $2\pi/2 = \pi$ and frequency $1/\pi \approx 0.3$. The amplitude of the oscillations is

$$|K| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.385$$

and the phase shift is

$$\phi = \frac{\arctan\left(\frac{-5}{2}\right)}{2} \approx \frac{-1.2}{2} = -0.6.$$

In fact, we can write the formula for the function $F$ (exactly) as

$$F(t) = \sqrt{29} \cos\left(2t - \arctan\left(\frac{-5}{2}\right)\right)$$

or (approximately) as

$$F(t) = 5.385 \cos (2t + 1.2)$$

or as

$$F(t) = 5.385 \cos (2(t + 0.6))$$

and we see that the graph of $F$ is the graph of $y = \cos (2t)$ amplified by a factor of 5.385 and shifted to the left by about 0.6 units.

**Exercise 5** A graph of $F(t) = -3\cos\left(\frac{1}{2}t\right) + 5\sin\left(\frac{1}{2}t\right)$ is shown below.
Write the formula for $F$ in the form $F(t) = K \cos(\omega t - \phi)$ and determine the period, frequency, amplitude, and phase shift for this function.

4 Solving the Forced Equation with Periodic Forcing

We now consider the problem of finding a particular solution of the forced equation with periodic forcing:

$$\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = \cos(\omega t) \quad (3)$$

with given parameters $p \geq 0$, $q > 0$, and $\omega > 0$. Note that we are restricting our attention to a forcing function ($\cos(\omega t)$) that has amplitude 1 and no phase shift. As will be seen, we will be able to handle more general forcing functions (sines and cosines with any amplitude or phase shift) once we understand how to find a solution to equation (3).

We claim that (in most circumstances) equation (3) has a solution of the form

$$F(t) = a \cos(\omega t) + b \sin(\omega t).$$

We can show that this is correct once we find the right values of $a$ and $b$. Keep the comment “in most circumstances” in mind. In the calculations that we are about to do, we will do some division operations. As we know, division is okay as long as we are not dividing by zero. In the calculations that follow, we will assume that when we are never dividing by zero. Then, when we are done, we will go back over our calculations and deal with cases where there may have been a danger of dividing by zero.
Defining the function $F$ as

$$F(t) = a \cos(\omega t) + b \sin(\omega t),$$

we observe that

$$\frac{dF}{dt} = -a \omega \sin(\omega t) + b \omega \cos(\omega t)$$

and

$$\frac{d^2F}{dt^2} = -a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t).$$

If $F$ is to be a solution of equation (3), then we must have

$$\begin{align*}
(-a\omega^2 \cos(\omega t) - b\omega^2 \sin(\omega t)) \\
+ p (-a \omega \sin(\omega t) + b \omega \cos(\omega t)) \\
+ q (a \cos(\omega t) + b \sin(\omega t)) \\
= \cos(\omega t) \text{ for all } t.
\end{align*}$$

This gives us

$$(-a\omega^2 + bp\omega + aq) \cos(\omega t) + (-b\omega^2 + q\omega b) \sin(\omega t) = \cos(\omega t)$$

or, equivalently,

$$((q - \omega^2) a + p\omega b) \cos(\omega t) + ((q - \omega^2) b - p\omega a) \sin(\omega t) = \cos(\omega t).$$

Since the above equation must be true for all real numbers $t$, it must be true when $t = 0$. This gives us

$$(q - \omega^2) a + p\omega b = 1.$$ 

Using $t = \pi/(2\omega)$, we obtain

$$-p\omega a + (q - \omega^2) b = 0.$$ 

Multiplying both sides of the first equation by $p\omega$, and multiplying both sides of the second equation by $q - \omega^2$ gives

$$p\omega (q - \omega^2) a + (p\omega)^2 b = p\omega$$

and

$$-p\omega (q - \omega^2) a + (q - \omega^2)^2 b = 0.$$
Adding these two equations gives us
\[
\left( (p\omega)^2 + (q - \omega^2)^2 \right) b = p\omega
\]
or
\[
b = \frac{p\omega}{(p\omega)^2 + (q - \omega^2)^2}.
\]
Having found \( b \), we can now determine that
\[
a = \frac{q - \omega^2}{(p\omega)^2 + (q - \omega^2)^2}.
\]
We thus claim that a solution of equation (3) is
\[
F(t) = \frac{q - \omega^2}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t) + \frac{p\omega}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t).
\] (4)
Let us check that this correct: Defining \( F \) as above, we have
\[
\frac{dF}{dt} = \frac{-\omega (q - \omega^2)}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t) + \frac{p\omega^2}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t)
\]
and
\[
\frac{d^2 F}{dt^2} = \frac{-\omega^2 (q - \omega^2)}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t) - \frac{p\omega^3}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t)
\]
and
\[
\frac{d^2 F}{dt^2} + \frac{p}{dt} + qF
\]
\[
= \frac{-\omega^2 (q - \omega^2)}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t) - \frac{p\omega^3}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t)
\]
\[
+ \frac{-p\omega (q - \omega^2)}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t) + \frac{p^2 \omega^2}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t)
\]
\[
+ \frac{q (q - \omega^2)}{(p\omega)^2 + (q - \omega^2)^2}\cos(\omega t) + \frac{pq\omega}{(p\omega)^2 + (q - \omega^2)^2}\sin(\omega t)
\]
\[
= -\omega^2 (q - \omega^2) + p^2 \omega^2 + q (q - \omega^2)\cos(\omega t)
\]
\[
+ \left( \frac{-p\omega^3 - p\omega (q - \omega^2) + pq\omega}{(p\omega)^2 + (q - \omega^2)^2} \right) \sin(\omega t)
\]
\[
= \cos(\omega t).
\]
The above computation shows that the function (4) is indeed a solution of the differential equation (3).

Observe that the formula for $F$ can be written as

$$F(t) = \frac{1}{(p\omega)^2 + (q - \omega^2)^2} \left( (q - \omega^2) \cos(\omega t) + p\omega \sin(\omega t) \right). \quad (5)$$

Assuming that $q - \omega^2 \neq 0$, we can use Lemma 2 to conclude that

$$(q - \omega^2) \cos(\omega t) + p\omega \sin(\omega t) = K \cos(\omega t - \phi) \quad (6)$$

where

$$K = \frac{|q - \omega^2| \sqrt{(p\omega)^2 + (q - \omega^2)^2}}{q - \omega^2}$$

and

$$\phi = \arctan \left( \frac{p\omega}{q - \omega^2} \right).$$

By observing that

$$K^2 = (p\omega)^2 + (q - \omega^2)^2,$$

we can conclude that

$$F(t) = \frac{1}{K} \cos(\omega t - \phi)$$

where $K$ and $\phi$ are as defined above. Thus, the function $F$ has phase shift $\phi/\omega$ and amplitude

$$\left| \frac{1}{K} \right| = \frac{1}{\sqrt{(p\omega)^2 + (q - \omega^2)^2}}.$$

Now let us consider the case that $q - \omega^2 = 0$. In this case, we have

$$F(t) = \frac{1}{p\omega} \sin(\omega t) = \frac{1}{p\omega} \cos \left( \omega t - \frac{\pi}{2} \right),$$

showing that $F$ has phase shift $\pi/(2\omega)$ and amplitude $1/(p\omega)$.

All of the above discussion relies on the assumption that either $p\omega \neq 0$ or $q - \omega^2 \neq 0$. Note that if at least one of $p\omega$ or $q - \omega^2$ is not zero, then $(p\omega)^2 + (q - \omega^2) > 0$ and we are thus justified in dividing by this quantity and in taking its square root. In particular, if $p > 0$ (meaning that the harmonic oscillator has damping present), then $(p\omega)^2 + (q - \omega^2) > 0$ (because we are also assuming that $\omega > 0$). We have therefore found a complete solution to our problem in the case that damping is present. We summarize as follows:
Theorem 6 Consider the differential equation

\[
\frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = \cos(\omega t)
\]

where \( p \geq 0, \, q > 0, \) and \( \omega > 0. \) Also, assume that either \( p > 0 \) or \( q-\omega^2 \neq 0. \)

Under the above assumptions, a solution of this differential equation is

\[
F(t) = \frac{q - \omega^2}{(p\omega)^2 + (q - \omega^2)^2} \cos(\omega t) + \frac{p\omega}{(p\omega)^2 + (q - \omega^2)^2} \sin(\omega t).
\]

Furthermore, \( F \) has amplitude

\[
\frac{1}{\sqrt{(p\omega)^2 + (q - \omega^2)^2}}
\]

and phase shift \( \phi/\omega \) where

\[
\phi = \begin{cases} 
\arctan\left(\frac{p\omega}{q-\omega^2}\right) & \text{if } q - \omega^2 \neq 0 \\
\frac{\pi}{2} & \text{if } q - \omega^2 = 0
\end{cases}
\]

Theorem 6 shows us how to find a particular solution of the periodically-forced harmonic oscillator differential equation if \( p > 0 \) (meaning that damping is present) or if \( q \neq \omega^2 \) (even if damping is not present). We must still consider the case in which \( p = 0 \) (meaning that damping is not present) and \( q = \omega^2. \) This is the special case in which we get the phenomenon known as resonance.

Before proceeding to study the case of resonance, let us make some observations about the conclusions of Theorem 6: First, suppose that \( p > 0 \) (meaning that damping is present). In this case, the forced response has amplitude

\[
\frac{1}{\sqrt{(p\omega)^2 + (q - \omega^2)^2}}
\]

It can be seen from this formula for the amplitude that if \( p \) is very large or \( \omega \) is very large, then the amplitude of the forced response is very small. Next, suppose that \( p = 0 \) (meaning that no damping is present) and that \( q \neq \omega^2: \) In this case, the forced response has amplitude

\[
\frac{1}{|q - \omega^2|}.
\]
and we see that if \( q \) is very close to \( \omega^2 \), then the amplitude of the forced response is very large. This is what we call the “near–resonant” case.

**Example 7** Consider the undamped, forced harmonic oscillator differential equation

\[
\frac{d^2y}{dt^2} + 4y = \cos(t).
\]

A particular solution of this equation is

\[
F(t) = \frac{1}{3}\cos(t).
\]

The general solution is

\[
y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{3}\cos(t).
\]

Suppose that we wish to find the particular solution that satisfies initial conditions \( y(0) = y'(0) = 0 \). Since

\[
y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) - \frac{1}{3}\sin(t),
\]

we must solve

\[
c_1 + \frac{1}{3} = 0
\]

\[
2c_2 = 0.
\]

This gives us

\[
c_1 = -\frac{1}{3}
\]

\[
c_2 = 0
\]

and the particular solution

\[
y(t) = -\frac{1}{3}\cos(2t) + \frac{1}{3}\cos(t),
\]

whose graph is shown below.
Example 8 In the previous example, we had \( q = 4 \) and \( \omega = 1 \) so \( q - \omega^2 = 3 \) is fairly large, meaning that the forced response has fairly small amplitude (1/3). Let us see what happens if we choose \( \omega \) such that \( \omega^2 \) is closer to \( q \). For example consider

\[
\frac{d^2 y}{dt^2} + 4y = \cos(1.9t).
\]

In this case, \( q - \omega^2 = 4 - (1.9)^2 = 0.39 \) and the forced response is

\[
F(t) = \frac{1}{0.39} \cos(1.9t).
\]

This forced response has amplitude \( 1/0.39 \approx 2.5641 \).

The general solution of the above differential equation is

\[
y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{0.39} \cos(1.9t).
\]

Suppose that we wish to find the particular solution that satisfies initial conditions \( y(0) = y'(0) = 0 \). Since

\[
y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{1.9}{0.39} \sin(1.9t),
\]

we must solve

\[
c_1 + \frac{1}{0.39} = 0
\]

\[
2c_2 = 0.
\]
This gives us

\[ c_1 = -\frac{1}{0.39} \]
\[ c_2 = 0 \]

and the particular solution

\[ y(t) = -\frac{1}{0.39} \cos(2t) + \frac{1}{0.39} \cos(1.9t) \],

whose graph is shown below.

Forced harmonic oscillators are fascinating, are they not?

5 The Case of Resonance

We now consider the differential equation

\[ \frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t) \]

where \( p = 0 \) and \( q = \omega^2 \). Thus, the differential equation that we are considering is

\[ \frac{d^2y}{dt^2} + \omega^2 y = \cos(\omega t) \]

with parameter \( \omega > 0 \).

We claim that this differential equation has a solution of the form

\[ F(t) = t(a \cos(\omega t) + b \sin(\omega t)) \].
As before, we determine what $a$ and $b$ must be in order to make this be true:

Letting $F$ be as defined above, we obtain

$$\frac{dF}{dt} = t (-\omega a \sin (\omega t) + \omega b \cos (\omega t)) + (a \cos (\omega t) + b \sin (\omega t))$$

$$= (a + \omega b) \cos (\omega t) + (-\omega t a + b) \sin (\omega t)$$

and

$$\frac{d^2F}{dt^2} = - (\omega a + \omega^2 tb) \sin (\omega t) + \omega b \cos (\omega t)$$

$$+ (\omega^2 t a + \omega b) \cos (\omega t) - \omega a \sin (\omega t)$$

$$= (-\omega^2 t a + 2 \omega b) \cos (\omega t) + (-2 \omega a - \omega^2 tb) \sin (\omega t).$$

If $F$ is to be a solution, we must have

$$( -\omega^2 t a + 2 \omega b) \cos (\omega t) + (-2 \omega a - \omega^2 tb) \sin (\omega t)$$

$$+ \omega^2 (t a \cos (\omega t) + t b \sin (\omega t))$$

$$= \cos (\omega t) \text{ for all real numbers } t.$$

Simplification of the above equation gives us

$$2 \omega b \cos (\omega t) - 2 \omega a \sin (\omega t) = \cos (\omega t).$$

Setting $t = 0$ yields

$$2 \omega b = 1$$

and setting $t = \pi / (2 \omega)$ yields

$$-2 \omega a = 0.$$

We conclude that $a = 0$ and $b = 1 / (2 \omega)$.

Let us check that

$$F(t) = \frac{1}{2 \omega} t \sin (\omega t)$$

is a solution of our differential equation: First note that

$$\frac{dF}{dt} = \frac{1}{2 \omega} (\omega t \cos (\omega t) + \sin (\omega t))$$
and
\[
\frac{d^2F}{dt^2} = \frac{1}{2\omega} \left( -\omega^2 t \sin(\omega t) + \omega \cos(\omega t) + \omega \cos(\omega t) \right)
= \frac{1}{2\omega} \left( -\omega^2 t \sin(\omega t) + 2\omega \cos(\omega t) \right).
\]
Thus
\[
\frac{d^2F}{dt^2} + \omega^2 F = \frac{1}{2\omega} \left( -\omega^2 t \sin(\omega t) + 2\omega \cos(\omega t) \right) + \omega^2 \left( \frac{1}{2\omega} t \sin(\omega t) \right)
= \cos(\omega t)
\]
showing that $F$ is indeed a solution.

We remark that since
\[
\sin(\omega t) = \cos \left( \omega t - \frac{\pi}{2} \right),
\]
we can also write the formula for $F$ as
\[
F(t) = \frac{1}{2\omega} t \cos \left( \omega t - \frac{\pi}{2} \right).
\]
This forced response has a phase shift of $\pi/(2\omega)$ and an amplitude that grows linearly as time goes on. If $\omega$ is very large, then the amplitude grows slowly; whereas, if $\omega$ is small, then the amplitude grows quickly.

**Example 9** Consider the undamped, forced harmonic oscillator differential equation
\[
\frac{d^2y}{dt^2} + 4y = \cos(2t).
\]
A particular solution of this equation (whose graph is shown below) is
\[
F(t) = \frac{1}{4} t \sin(2t).
\]
The general solution is

\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{4}t \sin(2t). \]

Suppose that we wish to find the particular solution that satisfies initial conditions \( y(0) = y'(0) = 0 \). Since

\[ y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{1}{2}t \cos(2t) + \frac{1}{4} \sin(2t), \]

we must solve

\[
\begin{align*}
    c_1 &= 0 \\
    2c_2 &= 0.
\end{align*}
\]

This gives us

\[
\begin{align*}
    c_1 &= 0 \\
    c_2 &= 0
\end{align*}
\]

and the particular solution is

\[ y(t) = \frac{1}{4}t \sin(2t) \]

whose graph is shown above.
6 More General Periodic Forcing Functions

Now that we know how to solve and study solutions of the forced harmonic oscillator equation

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = \cos(\omega t), \quad (7) \]

we would like to be able to solve harmonic oscillator equations with more general sinusoidal forcing functions. In particular, we would like to be able to solve

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = A \cos(\omega t - \theta) \quad (8) \]

where \( A, \omega, \) and \( \theta \) are given constants. Fortunately, it is easy to find a solution of equation (8) once we have found a solution of equation (7). In particular, suppose that \( F \) is a solution of equation (7) and define

\[ G(t) = AF \left( t - \frac{\theta}{\omega} \right). \]

Then

\[
G''(t) + pG'(t) + qG(t) = AF'' \left( t - \frac{\theta}{\omega} \right) + pAF' \left( t - \frac{\theta}{\omega} \right) + qAF \left( t - \frac{\theta}{\omega} \right) \\
= A \left( F'' \left( t - \frac{\theta}{\omega} \right) + pF' \left( t - \frac{\theta}{\omega} \right) + qF \left( t - \frac{\theta}{\omega} \right) \right) \\
= A \cos \left( \omega \left( t - \frac{\theta}{\omega} \right) \right) \\
= A \cos(\omega t - \theta)
\]

which shows that \( G \) is a solution of equation (8). In summary, if \( F \) is a solution of the forced equation

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = \cos(\omega t), \]

then \( G(t) = AF \left( t - \frac{\theta}{\omega} \right) \) is a solution of the forced equation

\[ \frac{d^2 y}{dt^2} + p \frac{dy}{dt} + qy = A \cos(\omega t - \theta). \]
Example 10  Solve the forced equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 3 \cos (t - 5). \]

Solution 11  First, we note that the general solution of the unforced equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 0 \]
is
\[ N(t) = c_1 e^{-3t} + c_2 e^{-2t}. \]

Next, we consider the forced equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = \cos(t). \]

Since \( p = 5, q = 6, \omega = 1, q - \omega^2 = 5, \) and \((p\omega)^2 + (q - \omega^2)^2 = 50,\) we see by equation (4) that a particular solution of this forced equation is
\[ F(t) = \frac{1}{10} \cos(t) + \frac{1}{10} \sin(t). \]

Finally, we consider the forced equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 3 \cos (t - 5). \]

A particular solution of this equation is
\[ G(t) = AF \left( t - \frac{\theta}{\omega} \right) \]
\[ = 3 \left( \frac{1}{10} \cos (t - 5) + \frac{1}{10} \sin (t - 5) \right). \]

In conclusion, the general solution of the forced equation
\[ \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 3 \cos (t - 5) \]
is
\[ y(t) = c_1 e^{-3t} + c_2 e^{-2t} + \frac{3}{10} \left( \cos (t - 5) + \sin (t - 5) \right). \]
Example 12 Solve the forced equation

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 2 \sin (4t - \pi) . \]

Solution 13 First we use the trigonometric identity

\[ \sin (\alpha) = \cos \left( \alpha - \frac{\pi}{2} \right) \]

to obtain

\[ 2 \sin (4t - \pi) = 2 \cos \left( 4t - \pi - \frac{\pi}{2} \right) . \]

Thus, we can rewrite our problem as

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 2 \cos \left( 4t - \frac{3\pi}{2} \right) . \]

Thus, \( A = 2, \omega = 4, \) and \( \theta = \frac{3\pi}{2} . \)

The general solution of the unforced equation,

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 0 \]

is

\[ N (t) = c_1 e^{-3t} + c_2 e^{-2t} . \]

Also, since \( p = 5, q = 6, q - \omega^2 = -10, \) and \( (p\omega)^2 + (q - \omega^2)^2 = 500, \) we see a solution of the forced equation

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = \cos (4t) \]

is

\[ F (t) = \frac{-1}{50} \cos (4t) + \frac{1}{25} \sin (4t) . \]

This means that a solution of the forced equation

\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 2 \cos \left( 4t - \frac{3\pi}{2} \right) \]
\[ G(t) = AF \left( t - \theta \right) \]
\[ = 2 \left( \frac{-1}{50} \cos \left( 4 \left( t - \frac{3\pi}{8} \right) \right) + \frac{1}{25} \sin \left( 4 \left( t - \frac{3\pi}{8} \right) \right) \right) \]
\[ = -\frac{1}{25} \cos \left( 4t - \frac{3\pi}{2} \right) + \frac{2}{25} \sin \left( 4t - \frac{3\pi}{2} \right). \]

In conclusion, the general solution of the forced equation
\[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y = 2 \cos \left( 4t - \frac{3\pi}{2} \right) \]
is
\[ y(t) = c_1e^{-3t} + c_2e^{-2t} + \frac{1}{25} \left( 2 \sin \left( 4t - \frac{3\pi}{2} \right) - \cos \left( 4t - \frac{3\pi}{2} \right) \right). \]