Foundations of Game Theory:

Reading: “Game Theory” (ECON 4550 Coursepak, Page 95)

Game Theory – study of decision making settings in which the outcome for each decision maker depends upon not only their own actions, but also upon the actions of other decision makers

• Within economics, Game Theory is very useful for analyzing the behavior of firms in the “intermediate market structures” between Monopoly and Perfect Competition

Three basic elements of any game:
1. Players – decision makers whose behavior is to be analyzed
2. Strategies – the different options or courses of action from which a player is able to choose
3. Payoffs – numerical measures of the desirability of every possible outcome which could arise as a result of the strategies chosen by the players

Example:

Players: Shell and Valero
Strategies: probably think of them choosing price
Payoffs: some function of profit or market share

“Payoff” of each “player” depends upon not only their own “strategy,” but also upon the strategy of their rival => appropriate to analyze the situation as a “game”
Possible Environments…

1. **Complete Information** versus **Incomplete Information**
   - **Complete Information** – an environment in which every player knows all of the strategies available to all players and the resulting payoff for all players at each possible outcome
   - **Incomplete Information** – an environment in which at least one of the players does not know all of the information that would be potentially relevant for making a decision at some point in the game
     - e.g., perhaps the other player can be of one of two different “types,” and I don’t know for certain which “type” he is

2. **Sequential Move Games** versus **Simultaneous Move Games**
   - **Sequential Move Game** – a game in which players make their decisions in sequence, with the choices made in the past being observable by all players when a present decision is being made
     - e.g., tic-tac-toe, Chess, Yahtzee
   - **Simultaneous Move Game** – a game in which players must each choose their strategies without being able to observe the strategy chosen by others (either the players choose their strategies at the same time, or it is as if they choose them at the same time)
     - e.g., Rock-Paper-Scissors
3. **One-Shot Games versus Repeated Games**
   - **One-Shot Game** – a game that is played between the same players only one time
   - **Repeated Game** – a game that is played between the same players more than one time

4. **Cooperative Games versus Non-Cooperative Games**
   - **Cooperative Game** – a game in which players can enter into binding agreements before the start of the game
   - **Non-Cooperative Game** – a game in which players cannot enter into binding agreements before the start of the game
     - in a non-cooperative environment, the choice of strategy by a player is fundamentally based upon what actions are in the self-interest of the player
     - in a non-cooperative game, players may still try to “talk ahead of time” to co-ordinate their actions and reach an agreement on how the game will be played, but such agreements (or commitments to play) are not binding => you should only expect another player to honor the agreement if his promise/threat is credible
       - **Credible promise/threat** – an announcement to behave in a certain way is credible if the stated action is in the best interest of the decision maker
         - e.g., “If you don’t eat your broccoli, you won’t get any ice cream…”
       - **Non-Credible promise/threat** – an announcement to behave in a certain way is non-credible if the stated action is not in the best interest of the decision maker
         - e.g., “If you don’t eat your broccoli, I’m going to cut your hand off…”
We will analyze...
- Non-Cooperative games with Complete Information
- both Simultaneous and Sequential Move games
- both One-Shot and Repeated games

Illustration of a One-Shot, Simultaneous Move game by way of a payoff matrix...
- Consider two firms operating in a market, each of which must choose either a “high” or “low” price
- Suppose the payoffs for the two firms at the four possible outcomes are summarized by the “payoff matrix” below

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Firm B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High “Price B”</td>
</tr>
<tr>
<td>High “Price A”</td>
<td>120 , 80</td>
</tr>
<tr>
<td>Low “Price A”</td>
<td>144 , 40</td>
</tr>
</tbody>
</table>

- In each cell the first number specifies the payoff of “player 1” (the player whose strategies are specified by each distinct row) and the second number specifies the payoff of “player 2” (the player whose strategies are specified by each distinct column)
- Each player gets to choose their own strategy, but has no control over the strategy chosen by their rival (i.e., “player 1” gets to choose the “row,” while “player 2” gets to choose the “column”)


Our initial aim will always be to identify a Nash Equilibrium for the game…

**Nash Equilibrium** – a set of strategies, one for each player, that are stable in the sense that no player could increase his own payoff by choosing a different strategy, given the strategies chosen by the other players

- very often, a Nash Equilibrium will serve as a reasonable prediction of play, since at the equilibrium strategies every player is behaving in a way that is in his own self-interest (given the behavior of others)

Named after the game theorist John F. Nash, Jr. (1928-2015; shared the Nobel Prize in Economics in 1994; subject of the 2001 Russell Crowe movie *A Beautiful Mind*)
To identify a Nash Equilibrium, we must systematically address the question “What should each player do?”

- Let us first examine the choice of “Firm A”…
  - “Firm A” has no control over the choice of “Firm B,” and further “Firm A” does not know what “Firm B” will necessarily do, but…
  - However, “Firm A” could determine what its own best choice would be for each of the things “Firm B” could possibly do
  - Further (if necessary), “Firm A” could also try to figure out what “Firm B” will do (based upon “what is best for ‘Firm B’ to do”)
- From the perspective of “Firm A”…
  - If “Firm B” were to choose “High Price B,” then “Firm A” would want to choose… “Low Price A,” since 144 > 120
  - If “Firm B” were to choose “Low Price B,” then “Firm A” would want to choose… “Low Price A,” since 84 > 64
**Best Reply** – for “Player 1” a strategy is a best reply to a chosen strategy of “Player 2” if the strategy in question gives “Player 1” a greater payoff than any other available strategy.

Note: a set of strategies is a Nash Equilibrium if and only if each player is choosing a strategy that is a “best reply” to the strategies being played by others…

- In the game above, for “Firm A”:
  - “Low Price A” is a best reply to a choice of “High Price” by “Firm B”
  - “Low Price A” is a best reply to a choice of “Low Price” by “Firm B”
  - “Low Price A” is a best reply to anything “B” can do

*Illustrate the “Best Replies” by drawing “arrows” in the payoff matrix as follows:*

<table>
<thead>
<tr>
<th>Firm A</th>
<th>Firm B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>High “Price A”</td>
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</tr>
<tr>
<td>Low “Price A”</td>
<td>144 , 40</td>
</tr>
</tbody>
</table>

**Dominant Strategy** – for ‘Player 1’ a strategy is a dominant strategy if it is (strictly) a best reply to all available strategies of ‘Player 2’

“**Strategic Rule of Thumb #1**” – if you have a dominant strategy, use it
• Note that in this example, “Firm B” also has a dominant strategy. From the perspective of “Firm B”…
   If “Firm A” were to choose “High Price A,” then “Firm B” would want to choose… “Low Price B,” since $96 > 80$
   If “Firm A” were to choose “Low Price A,” then “Firm B” would want to choose… “Low Price B,” since $56 > 40$

• The unique Nash Equilibrium is for “Firm A” to choose “Low Price A” and for “Firm B” to choose “Low Price B.” As a result, “Firm A” realizes a payoff of (84) and “Firm B” realizes a payoff of (56)

<table>
<thead>
<tr>
<th></th>
<th>High “Price B”</th>
<th>Low “Price B”</th>
</tr>
</thead>
<tbody>
<tr>
<td>High “Price A”</td>
<td>120, 80</td>
<td>60, 96</td>
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• The unique Nash Equilibrium is for “Firm A” to choose “Low Price A” and for “Firm B” to choose “Low Price B.” As a result, “Firm A” realizes a payoff of (84) and “Firm B” realizes a payoff of (56)
What if each firm instead had a choice of three different prices, as follows?

<table>
<thead>
<tr>
<th>Firm A</th>
<th>High “Price A”</th>
<th>Mid “Price A”</th>
<th>Low “Price A”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>120 , 80</td>
<td>148 , 42</td>
<td>144 , 40</td>
</tr>
<tr>
<td></td>
<td>80 , 100</td>
<td>96 , 64</td>
<td>106 , 44</td>
</tr>
<tr>
<td></td>
<td>60 , 96</td>
<td>74 , 72</td>
<td>84 , 56</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Firm B</th>
<th>High “Price B”</th>
<th>Mid “Price B”</th>
<th>Low “Price B”</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>120 , 80</td>
<td>80 , 100</td>
<td>60 , 96</td>
</tr>
<tr>
<td>Mid</td>
<td>148 , 42</td>
<td>96 , 64</td>
<td>74 , 72</td>
</tr>
<tr>
<td>Low</td>
<td>144 , 40</td>
<td>106 , 44</td>
<td>84 , 56</td>
</tr>
</tbody>
</table>

Does either player have a “Dominant Strategy”?
- “Firm A” does not, since
  - “Low Price A” is a best reply to a choice of either “Mid Price B” or “Low Price B”
  - But “Mid Price A” is the best reply to “High Price B”
- Similarly, “Firm B” does not, since
  - “Low Price B” is a best reply to a choice of either “Mid Price A” or “Low Price A”
  - But “Mid Price B” is the best reply to “High Price A”

Dominated Strategy – for ‘Player 1’ a strategy is a dominated strategy there is some other available strategy that gives ‘Player 1’ a (strictly) higher payoff than the strategy in question, for all available strategies of ‘Player 2’

- In the ‘3x3 game’ above…
  - “High Price A” is dominated for “Firm A” by both “Mid Price A” and “Low Price A”
  - Similarly, “High Price B” is dominated for “Firm B” by both “Mid Price B” and “Low Price B”

“Strategic Rule of Thumb #2” – if you have a dominated strategy, do not use it
“Strategic Rule of Thumb #3” – if your rival has a dominant strategy, expect her to use it

“Strategic Rule of Thumb #4” – if your rival has a dominated strategy, expect her to never use it

• Thus, not only will “Firm A” never choose “High Price A” and “Firm B” will never choose “High Price B,” but
  ▪ “Firm B” can reasonably infer that “Firm A” will never choose “High Price A” and
  ▪ “Firm A” can reasonably infer that “Firm B” will never choose “High Price B”

• the row and column crossed out above are each irrelevant
  ▪ even though neither player had a dominant strategy in the “full game,” in the “reduced game” (below) it is as if each player does…

• This game has a unique Nash Equilibrium for which “Firm A” chooses “Low Price A” and “Firm B” chooses “Low Price B”
• The equilibrium in the game above was determined by a procedure known as “Iterated Elimination of Dominated Strategies”

**Iterated Elimination of Dominated Strategies** – a process by which strategies that cannot be part of a Nash Equilibrium are eliminated from consideration, by

1. discarding any strategy that is dominated for a player
2. examining the “reduced game” and discarding any strategy that is dominated for a player in the “reduced game”
3. repeating “Step 2” until no strategies for any player can be discarded

• Any strategy that is “discarded” during the process of Iterated Elimination of Dominated Strategies can never be a strategy that is used by a player at a Nash Equilibrium
• This process might lead to the identification of a unique Nash Equilibrium (as in the example above)
• It may lead to the identification of a unique Nash Equilibrium in even more complex games (as illustrated by the next example)
• But, it will not always be possible to whittle the game down to “one unique strategy for each player” (as illustrated by the example following the next example)
Example of solving a larger game by applying “IEDS”:

<table>
<thead>
<tr>
<th>Player 2</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10, 5</td>
<td>2, 12</td>
<td>1, 0</td>
<td>3, 6</td>
<td>4, 14</td>
</tr>
<tr>
<td>B</td>
<td>8, 3</td>
<td>4, 8</td>
<td>3, 0</td>
<td>1, 4</td>
<td>7, 3</td>
</tr>
<tr>
<td>C</td>
<td>6, 4</td>
<td>5, 6</td>
<td>4, 1</td>
<td>6, 8</td>
<td>2, 7</td>
</tr>
<tr>
<td>D</td>
<td>4, 7</td>
<td>3, 8</td>
<td>2, 10</td>
<td>5, 9</td>
<td>5, 8</td>
</tr>
<tr>
<td>E</td>
<td>2, 2</td>
<td>4, 1</td>
<td>5, 4</td>
<td>0, 5</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

- Recognize that neither player has a dominant strategy
- Further, for “Player 1” none of the strategies are dominated
- However, for “Player 2” strategy “a” is dominated by “d” => “d” is not always the best choice for “Player 2,” but it is always better than “a” (for anything that “Player 1” could choose)
- Eliminate “a”…

<table>
<thead>
<tr>
<th>Player 2</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2, 12</td>
<td>1, 0</td>
<td>3, 6</td>
<td>4, 14</td>
</tr>
<tr>
<td>B</td>
<td>4, 8</td>
<td>3, 0</td>
<td>1, 4</td>
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</tr>
<tr>
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<td>4, 1</td>
<td>6, 8</td>
<td>2, 7</td>
</tr>
<tr>
<td>D</td>
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<td>2, 10</td>
<td>5, 9</td>
<td>5, 8</td>
</tr>
<tr>
<td>E</td>
<td>4, 1</td>
<td>5, 4</td>
<td>0, 5</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

- In this “reduced game,” for “Player 1” the strategy of “A” is now dominated by “D” => eliminate “A”
In this “reduced game,” for “Player 2” the strategy of “e” is now dominated by “d” => eliminate “e”

In this “reduced game,” for “Player 1” the strategies of “B” and “D” are dominated by “C” => eliminate “B” and “D”

In this “reduced game,” for “Player 2” the strategies of “b” and “c” are dominated by “d” => eliminate “b” and “c”
• In this “reduced game,” for “Player 1” the strategy of “E” is dominated by “C” ⇒ eliminate “E”

• Via Iterated Elimination of Dominated Strategies, we have identified the strategy pair of “C” for “Player 1” and “d” for “Player 2” as the unique Nash Equilibrium of the game

Example for which “IEDS” cannot solve the game:
• But, not all games can be solved by this approach. Consider:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>8, 7</td>
</tr>
<tr>
<td>Bottom</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

• Neither player has a dominant strategy or a dominated strategy ⇒ cannot whittle away any strategies by IEDS
But, this game is simple enough that we can see (by drawing our “best response arrows”) that there are actually multiple Nash Equilibria

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<td>Left: 8, 7</td>
</tr>
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<td>Bottom</td>
<td>Left: 2, 1</td>
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**In a ‘2x2 game,’ what do the ‘best response arrows’ reveal?**

1. Whether a player has or does not have a dominant strategy
   - If all the arrows for a player “point in the same direction,” then the strategy associated with either the row or column to which they all point is a dominant strategy (since this indicates that the “best reply” for the player is the same, regardless of the strategy chosen by her rival)
   - If all the arrows for a player do NOT “point in the same direction,” then the player does not have a dominant strategy (since this indicates that the “best reply” for the player depends upon the strategy chosen by her rival)

2. Whether a pair of strategies is or is not an equilibrium
   - If for a particular “cell” all the arrows “point inward,” then the pair of strategies leading to this cell is a Nash Equilibrium (since this indicates that no player could increase her own payoff by choosing a different strategy)
   - If for a particular “cell” any arrows “point outward,” then the pair of strategies leading to this cell is NOT a Nash Equilibrium (since this indicates that at least one player could increase her payoff by choosing a different strategy)
Returning attention to our initial example of…

Firm A

<table>
<thead>
<tr>
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<tr>
<td>High “Price A”</td>
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<td>144, 40 84, 56</td>
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- Since each player in this game has a dominant strategy (of “Low Price…”) it seems pretty clear that the equilibrium of “Low Price A” and “Low Price B” will be the outcome
- But, is this outcome “desirable” for the players?
- Recognize that if both players could instead choose “High Price…” then both “Player 1” and “Player 2” would be better off!
- This game is an example of a Prisoners’ Dilemma…

**Prisoner’s Dilemma** – a game in which every player has a dominant strategy (so that the game has a unique equilibrium characterized by all players using their dominant strategies), but in which there is some other outcome at which the payoff of every single player is higher than the equilibrium payoff

- The game above is a “Prisoner’s Dilemma” since each player would have a higher payoff if “Firm A” chose “High Price A” and “Firm B” chose “High Price B”
- But again, this “high price, high price” outcome in not stable, since each player has an individual incentive to “cheat” and “charge the lower price”
Thinking of this game as a model of “competition” between two firms, what these insights suggest is that the two firms could each do better if they could “collude” and each charge the “high price”…

**Collusion** – an effort by firms to coordinate their actions in an attempt to increase both “total industry profit” and “individual profit of every firm” (compared to the profit levels which would result without any such coordination)

The firms could do better by forming a cartel…

**Cartel** – a group of firms who attempt to engage in collusion, either openly/explicitly or tacitly/implicitly

“Tacit Collusion” (i.e., collusion in which firms are not “legally bound” to take the coordinated actions) is often “non-sustainable” since very often:

- each firm has an incentive to “cheat” on the agreement
- for example, if firms are attempting to collude by all charging the “monopoly price” and “splitting monopoly profits,” there is a great deal to be gained for “any one firm” by being “the one firm charging a low price” => firms face a “Prisoner’s Dilemma”

But, intuitively it seems as if the players may be able to resolve the dilemma by playing the game more than once…

**Repeated Prisoner’s Dilemma** – a standard prisoner’s dilemma game which confronts the same players repeatedly over time
Consider the following Prisoners’ Dilemma:

\[
\begin{array}{c|cc}
\text{Player 1} & \text{Cooperate} & \text{Defect} \\
\hline
\text{Cooperate} & 10, 10 & 2, 12 \\
\text{Defect} & 12, 2 & 4, 4 \\
\end{array}
\]

Suppose this game is repeated two times. Before the first play of the game, you tell your rival:

- in each period we should both choose “cooperate”
- I will choose cooperate in the first period, and then in the second period I’ll…
- …again choose “cooperate” if you chose “cooperate” in the first period
- …choose “defect” if you chose “defect” in the first period
- It would seem as if each player can get a total payoff of \((10)+(10)=(20)\) by going along with this plan
- If I instead choose “defect” in the first period to get a higher payoff of \((12)\), then you will punish me in the second period => I will choose “defect” again and get a payoff of \((4)\)
- \((4)+(12)=(16)\), which is less than \((20)\), so I should go along with the plan, right?
- Wait, wait, wait… What if I choose “cooperate” in the first period but then “defect” in the second! Now I get a payoff of \((10)+(12)=(22)\), which is better than \((20)\)! => your plan does not work
- In fact, once we get to the second period, each player has an incentive to choose “defect,” since at this point it is simply a “one shot Prisoners’ Dilemma” => as long as we both know for certain what period is last, we cannot maintain cooperation
But what if instead it’s a repeated game with an uncertain end time? Suppose that after each period we will play “one more time” with probability $p$?

- Recognize that in a repeated game, strategies can be very complex, since at any point in time a player can condition his choice on anything that has previously been observed.
- Examples of strategies in this repeated Prisoners’ Dilemma
  - always choose cooperate in every period
  - always choose defect in every period
  - choose cooperate in every “even period” and defect in every “odd period”
  - in each period play whatever strategy my opponent played last period (tit-for-tat)
  - always choose cooperate so long as my opponent has chosen cooperate in every previous period, but once my opponent chooses defect then choose defect from in every single future period (grim trigger)

If my opponent announces that she will play according to grim trigger, what is my best reply?

- If I choose cooperate in every period, then my expected payoff is: $\pi_C = 10 + 10p + 10p^2 + 10p^3 + 10p^4 + ...$
- If I instead choose defect in every period, then my expected payoff is: $\pi_D = 12 + 4p + 4p^2 + 4p^3 + 4p^4 + ...$
- Always choosing cooperate is better if and only if:
  $$\pi_C \geq \pi_D$$
  $$10 + 10p + 10p^2 + 10p^3 + 10p^4 + ... \geq 12 + 4p + 4p^2 + 4p^3 + 4p^4 + ...$$
  $$6p + 6p^2 + 6p^3 + 6p^4 + ... \geq 2$$
  $$p + p^2 + p^3 + p^4 + ... \geq \frac{2}{6} = \frac{1}{3}$$
What is $p + p^2 + p^3 + p^4 + \ldots$ equal to?

- Define $p + p^2 + p^3 + p^4 + \ldots = \sum_{i=1}^{\infty} p^i \equiv A$

- Recognize that $Ap = p^2 + p^3 + p^4 + p^5 + \ldots$

- Thus, $A - Ap = (p + p^2 + p^3 + \ldots) - (p^2 + p^3 + p^4 + \ldots)$

- That is, $A - Ap = p$

- It follows that $A(1 - p) = p \iff A = \frac{p}{1-p}$

For this game always choosing cooperate is a best reply to grim trigger (in which case repetition can sustain cooperation) so long as

$$\frac{p}{1-p} \geq \frac{1}{3}$$

$$\iff 3p \geq 1 - p$$

$$\iff 4p \geq 1$$

$$\iff p \geq \frac{1}{4}$$

So, “repeated interaction” may be able to resolve the dilemma (but only if players do not know which period will be the last)

It is also conceptually possible to resolve the dilemma by coming to a legal agreement to “cooperate” or “not defect”…
Cigarette Advertising as a Prisoners’ Dilemma:
Resolving the dilemma by “legal restrictions”…

- Studies have shown that in many industries (including cigarettes) advertising primarily ‘makes consumers switch brands,’ as opposed to ‘attracting new customers’
- Consider two cigarette manufacturers, each choosing between “advertising on TV” or “No TV ads”

<table>
<thead>
<tr>
<th></th>
<th>Philip Morris</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Advertise on TV</td>
<td>No TV ads</td>
</tr>
<tr>
<td>RJ</td>
<td>10, 20</td>
<td>40, 15</td>
</tr>
<tr>
<td>Reynolds</td>
<td>No TV ads</td>
<td>5, 50</td>
</tr>
</tbody>
</table>

- The simple fact that these firms were engaged in a “repeated prisoner’s dilemma” was not enough to “resolve the dilemma” prior to 1971. However, on January 2, 1971 they were able to “resolve the dilemma” and no longer advertise on TV
- Approximate advertising expenditures by cigarette manufacturers: $300 million in 1970; $240 in 1971
- Firms were able to realize larger profits in 1971 after “resolving the dilemma” => great news for cigarette manufacturers!
- How were they able to do this?
  - Starting on 1/2/71, there was a legal agreement which made it that firms would no longer advertise on TV
  - How could the Government “let this happen”!? 
  - The government was the one who “made it happen” => on 1/2/71 the “Public Health Cigarette Smoking Act” (which made it illegal to advertise cigarettes on TV and radio) went into effect
Do all one-shot games have a Nash Equilibrium?
• Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
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</tr>
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<tbody>
<tr>
<td>Top</td>
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<td>10, 1</td>
</tr>
<tr>
<td>Bottom</td>
<td>6, 3</td>
<td>2, 5</td>
</tr>
</tbody>
</table>

• Based upon our insights thus far, there does not appear to be a Nash Equilibrium! How should we expect players to behave?

“Mixed Extension” and “Nash’s Existence Theorem”:
• What would you do if you were playing this game?
• You would probably want your choice to be “seemingly random” from the point of view of your rival

Mixed Extension of a Game – an interpretation of a game in which the strategy choice of a player is allowed to be a “probability distribution” over their available “pure strategies”

• Let $q$ denote the probability with which “Player 2” chooses “Left” (so that $1-q$ denotes the probability with which “Player 2” chooses “Right”)
• Let $p$ denote the probability with which “Player 1” chooses “Top” (so that $1-p$ denotes the probability with which “Player 1” chooses “Bottom”)
• The “Best Reply” for each player can be illustrated by way of a “Best Reply Correspondence” (a Correspondence is a “multi-valued function”)
Consider the choice by “Player 1” when “Player 2” chooses “Left” with probability $q$ and “Right” with probability $1-q$ ...

- Player 1’s expected payoff from choosing “Top” is:
  \[ \pi^T_1 = 4q + 10(1-q) = 10 - 6q \]
- Player 1’s expected payoff from choosing “Bottom” is:
  \[ \pi^B_1 = 6q + 2(1-q) = 2 + 4q \]
- Thus, Player 1’s payoff is strictly greater from choosing “Top” as opposed to “Bottom” (in which case his “best reply” is $p = 1$) if and only if
  \[ \pi^T_1 > \pi^B_1 \]
  \[ \iff 10 - 6q > 2 + 4q \]
  \[ \iff 8 > 10q \]
  \[ \iff q < \frac{8}{10} = \frac{4}{5} = .8 \]
- Further, Player 1’s payoff is strictly greater from choosing “Bottom” as opposed to “Top” (in which case his “best reply” is $p = 0$) if and only if $q > .8$
- Finally, Player 1 would realize the exact same expected payoff from choosing “Top,” “Bottom,” or any randomization between the two (in which case any value of $0 \leq p \leq 1$ is a “best reply”) if and only if $q = .8$
- Visually, the Best Reply Correspondence of Player 1 can be illustrated as:
Similarly, considering the choice by “Player 2” when “Player 1” chooses “Top” with probability $p$ and “Bottom” with probability $1 - p$ …

- Player 2’s expected payoff from choosing “Left” is:
  $$\pi_2^L = 7p + 3(1 - p) = 3 + 4p$$

- Player 2’s expected payoff from choosing “Right” is:
  $$\pi_2^R = p + 5(1 - p) = 5 - 4p$$

- Thus, Player 2’s payoff is strictly greater from choosing “Left” as opposed to “Right” (in which case his “best reply” is $q = 1$) if and only if
  $$\pi_2^L > \pi_2^R$$
  $$\Leftrightarrow 3 + 4p > 5 - 4p$$
  $$\Leftrightarrow p > \frac{2}{8} = \frac{1}{4} = .25$$

- Further, Player 2’s payoff is strictly greater from choosing “Right” as opposed to “Left” (in which case his “best reply” is $q = 0$) if and only if $p < .25$

- Finally, Player 2 would realize the exact same expected payoff from choosing “Left,” “Right,” or any randomization between the two (in which case any value of $0 \leq q \leq 1$ is a “best reply”) if and only if $p = .25$

- Visually, the Best Reply Correspondence of Player 2 can be illustrated as:
Drawing the two correspondences in the same graph, we have:

- If Player 1 chooses \( p^* = 0.25 \) and Player 2 chooses \( q^* = 0.8 \), then each player is choosing a mixed strategy that is a best reply to the strategy being played by his rival \( \Rightarrow \) this pair of mixed strategies is a Nash Equilibrium!

**Nash’s Existence Theorem** – for every game with any finite number of players, each with a finite number of available pure strategies, there exists at least one Nash Equilibrium (potentially in “mixed strategies”)

- In practice, even if a player doesn’t “actually randomize,” there is often a benefit of doing your best to make it “look as if your behavior is random” (or “unpredictable”). For Example:
  - How well would a baseball pitcher do if “every other pitch was a curve ball and every other pitch was a fastball”?
  - How well would a football team do if “they always ran on 1st and 2nd down and always passed on 3rd down”?
Sequential Move Games:

- Recall, in a sequential move game, when a player is called upon to act, he can observe all of the choices that have been made previously in the game.
- Example: “Firm 2” currently operates in both “Market A” and “Market B,” while “Firm 1” currently only operates in “Market A” but is considering entry into “Market B.”
  - Suppose that “Firm 1” first chooses to either “Enter” or “Not Enter” this second market.
  - After observing this decision, “Firm 2” will choose to either “maintain collusion” or start a “price war.”
- This sequential interaction between the two players can be illustrated by a game tree (very similar to the decision trees that we discussed earlier in the course, except now: (i) decisions are being made by more than one person and (ii) we must specify payoffs for more than one person).
- Suppose:
• Can we identify a Nash Equilibrium for this game?
  - Recall, a Nash Equilibrium is: “a set of strategies, one for each player, that are stable in the sense that no player could increase his own payoff by choosing a different strategy, given the strategies chosen by the other players”

• Consider the following pair of strategies:
  - Firm 2 chooses “Maintain Collusion” if Firm 1 chooses “Don’t Enter Market B” and chooses “Price War” if Firm 1 chooses “Enter Market B” (i.e., Firm 1 threatens a price war if and only if Firm 2 enters…)
  - Firm 1 chooses “Don’t Enter Market B”
  - These pair of strategies can be illustrated as:

• This pair of strategies does fit our definition of a Nash Equilibrium (…check all possible “unilateral deviations”…), but it doesn’t seem reasonable! Why not?
• Think about the decision of Firm 2 if Firm 1 were to actually choose “Enter Market B”…
• The “threat” by Firm 1 to engage in a “price war” following entry is NOT CREDIBLE => thus, it does not seem reasonable to support an equilibrium on such a threat that would clearly not be carried out if Player 1 were to ever “call the bluff” of Player 2
• For sequential move games, a “refinement” of the Nash Equilibrium concept has been developed, in order to rule out such unreasonable equilibria => Subgame Perfect Nash Equilibrium

Subgame – a sequential move game of complete information with \( n \) decision nodes has \( n \) subgames, each consisting of a decision node along with all subsequent branches of the game tree

The game above consists of three subgames…
Subgame Perfect Nash Equilibrium – an equilibrium in which every decision of every player is optimal within all subgames of the larger game

- This refinement of the Nash Equilibrium concept rules out any equilibria that are supported by non-credible threats
- So long as each player has a different payoff at each terminal node of the game, there will be exactly one SPNE for the game
- This unique SPNE can be identified via “backward induction”
- First, consider the choice by “Firm 2” following each possible choice by “Firm 1”…
• Next, considering the initial choice by “Firm 1”...

• Unique SPNE consists of the following pair of strategies (note: we must specify the optimal choice at every decision node, even those that are not reached during the play of the game)
  - Firm 2 chooses “Maintain Collusion” if Firm 1 chooses “Don’t Enter Market B” and chooses “Maintain Collusion” if Firm 1 chooses “Enter Market B”
  - Firm 1 chooses “Enter Market B”

• This solution is illustrated most directly by drawing arrows along each relevant branch of the game tree (again, indicate exactly one choice after every decision node) and circling the payoff vector at the realized outcome