Economics of Strategy (ECON 4550)
“Foundations of Game Theory”

Reading: “Game Theory” (ECON 4550 Coursepak, Page 95)

Definitions and Concepts:

• Game Theory – study of decision making settings in which the outcome for each decision maker depends upon not only their own actions, but also upon the actions of other decision makers
  ▪ Within economics, Game Theory is very useful for analyzing the behavior of firms in the “intermediate market structures” between Monopoly and Perfect Competition

• Three basic elements of any game:
  1. Players – decision makers whose behavior is to be analyzed
  2. Strategies – the different options or courses of action from which a player is able to choose
  3. Payoffs – numerical measures of the desirability of every possible outcome which could arise as a result of the strategies chosen by the players

• Complete Information – an environment in which every player knows all of the strategies available to all players and the resulting payoff for all players at each possible outcome

• Incomplete Information – an environment in which at least one of the players does not know all of the information that would be potentially relevant for making a decision at some point in the game
  ▪ e.g., perhaps the other player can be of one of two different “types,” and I don’t know for certain which “type” he is

• Sequential Move Game – a game in which players make their decisions in sequence, with the choices made in the past being observable by all players when a present decision is being made

• Simultaneous Move Game – a game in which players must each choose their strategies without being able to observe the strategy chosen by others (either the players choose their strategies at the same time, or it is as if they choose them at the same time)

• One-Shot Game – a game that is played between the same players only one time

• Repeated Game – a game that is played between the same players more than one time
• **Cooperative Game** – a game in which players can enter into binding agreements before the start of the game

• **Non-Cooperative Game** – a game in which players cannot enter into binding agreements before the start of the game

• **Credible promise/threat** – an announcement to behave in a certain way is credible if the stated action is in the best interest of the decision maker

• **Non-Credible promise/threat** – an announcement to behave in a certain way is non-credible if the stated action is not in the best interest of the decision maker

• **Nash Equilibrium** – a set of strategies, one for each player, that are stable in the sense that no player could increase his own payoff by choosing a different strategy, given the strategies chosen by the other players
  ▪ very often, a Nash Equilibrium will serve as a reasonable prediction of play, since at the equilibrium strategies every player is behaving in a way that is in his own self-interest (given the behavior of others)
  ▪ Named after the game theorist John F. Nash, Jr. (1994 Nobel Prize in Economics)

• **Best Reply** – for “Player 1” a strategy is a best reply to a chosen strategy of “Player 2” if the strategy in question gives “Player 1” a greater payoff than any other available strategy
  ▪ **Note:** a set of strategies is a Nash Equilibrium if and only if each player is choosing a strategy that is a “best reply” to the strategies being played by others

• **Dominant Strategy** – for ‘Player 1’ a strategy is a dominant strategy if it is (strictly) a best reply to all available strategies of ‘Player 2’

• **“Strategic Rule of Thumb #1”** – if you have a dominant strategy, use it

• **Dominated Strategy** – for ‘Player 1’ a strategy is a dominated strategy there is some other available strategy that gives ‘Player 1’ a (strictly) higher payoff than the strategy in question, for all available strategies of ‘Player 2’

• **“Strategic Rule of Thumb #2”** – if you have a dominated strategy, do not use it

• **“Strategic Rule of Thumb #3”** – if your rival has a dominant strategy, expect her to use it

• **“Strategic Rule of Thumb #4”** – if your rival has a dominated strategy, expect her to never use it
• **Iterated Elimination of Dominated Strategies** – a process by which strategies that cannot be part of a Nash Equilibrium are eliminated from consideration, by
  1. discarding any strategy that is dominated for a player
  2. examining the “reduced game” and discarding any strategy that is dominated for a player in the “reduced game”
  3. repeating “Step 2” until no strategies for any player can be discarded
    ▪ Any strategy that is “discarded” during the process of Iterated Elimination of Dominated Strategies can never be a strategy that is used by a player at a Nash Equilibrium
    ▪ This process might lead to the identification of a unique Nash Equilibrium, even in seemingly complex games (i.e., ones in which players have several different available strategies)
    ▪ But, this process will not always whittle the game down to “one unique strategy for each player”

• **Prisoner’s Dilemma** – a game in which every player has a dominant strategy (so that the game has a unique equilibrium characterized by all players using their dominant strategies), but in which there is some other outcome at which the payoff of every single player is higher than the equilibrium payoff

• **Collusion** – an effort by firms to coordinate their actions in an attempt to increase both “total industry profit” and “individual profit of every firm” (compared to the profit levels which would result without any such coordination)

• **Cartel** – a group of firms who attempt to engage in collusion, either openly/explicitly or tacitly/implicitly

• **Repeated Prisoner’s Dilemma** – a standard prisoner’s dilemma game which confronts the same players repeatedly over time

• **Mixed Extension of a Game** – an interpretation of a game in which the strategy choice of a player is allowed to be a “probability distribution” over their available “pure strategies”

• **Nash’s Existence Theorem** – for every game with any finite number of players, each with a finite number of available pure strategies, there exists at least one Nash Equilibrium (potentially in “mixed strategies”)

• **Subgame** – a sequential move game of complete information with \( n \) decision nodes has \( n \) subgames, each consisting of a decision node along with all subsequent branches of the game tree

• **Subgame Perfect Nash Equilibrium** – an equilibrium in which every decision of every player is optimal within all subgames of the larger game
A simple “2x2 game” in which each player has a dominant strategy…

<table>
<thead>
<tr>
<th>Firm A</th>
<th>High “Price A”</th>
<th>Low “Price A”</th>
</tr>
</thead>
<tbody>
<tr>
<td>High “Price B”</td>
<td>120, 80</td>
<td>60, 96</td>
</tr>
<tr>
<td>Low “Price B”</td>
<td>144, 40</td>
<td>84, 56</td>
</tr>
</tbody>
</table>

In each cell the first number specifies the payoff of “player 1” (the player whose strategies are specified by each distinct row) and the second number specifies the payoff of “player 2” (the player whose strategies are specified by each distinct column).

Each player gets to choose their own strategy, but has no control over the strategy chosen by their rival (i.e., “player 1” gets to choose the “row,” while “player 2” gets to choose the “column”)

To identify a Nash Equilibrium, we must systematically address the question “What should each player do?”

Let us first examine the choice of “Firm A”…

- **“Firm A” has no control over the choice of “Firm B,”** and further “Firm A” does not know what “Firm B” will necessarily do, but…
- However, “Firm A” could determine what its own best choice would be for each of the things “Firm B” could possibly do
- Further (if necessary), “Firm A” could also try to figure out what “Firm B” will do (based upon “what is best for ‘Firm B’ to do”)

From the perspective of “Firm A”…

- If “Firm B” were to choose “High Price B,” then “Firm A” would want to choose “Low Price A,” since 144 > 120
- If “Firm B” were to choose “Low Price B,” then “Firm A” would want to choose “Low Price A,” since 84 > 64

In this game, for “Firm A”:

- “Low Price A” is a best reply to a choice of “High Price” by “Firm B”
- “Low Price A” is a best reply to a choice of “Low Price” by “Firm B”
- “Low Price A” is a best reply to anything “B” can do

Note that “Firm B” also has a dominant strategy. From the perspective of “Firm B”…

- If “Firm A” were to choose “High Price A,” then “Firm B” would want to choose “Low Price B,” since 96 > 80
- If “Firm A” were to choose “Low Price A,” then “Firm B” would want to choose “Low Price B,” since 56 > 40

The unique Nash Equilibrium is for “Firm A” to choose “Low Price A” and for “Firm B” to choose “Low Price B.” As a result, “Firm A” realizes a payoff of (84) and “Firm B” realizes a payoff of (56)
Solving a larger game by applying “Iterated Elimination of Dominated strategies”:

<table>
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<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
<tr>
<td>A</td>
<td>10, 5</td>
<td>2, 12</td>
<td>1, 0</td>
<td>3, 6</td>
<td>4, 14</td>
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<tr>
<td>B</td>
<td>8, 3</td>
<td>4, 8</td>
<td>3, 0</td>
<td>1, 4</td>
<td>7, 3</td>
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<tr>
<td>C</td>
<td>6, 4</td>
<td>5, 6</td>
<td>4, 1</td>
<td>6, 8</td>
<td>2, 7</td>
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<td>D</td>
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<td>5, 9</td>
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<td>E</td>
<td>2, 2</td>
<td>4, 1</td>
<td>5, 4</td>
<td>0, 5</td>
<td>1, 0</td>
</tr>
</tbody>
</table>

- Recognize that neither player has a dominant strategy
- Further, for “Player 1” none of the strategies are dominated
- However, for “Player 2” strategy “a” is dominated by “d” => “d” is not always the best choice for “Player 2,” but it is always better than “a” (for anything that “Player 1” could choose)
- Eliminate “a”…

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- In this “reduced game,” for “Player 1” the strategy of “A” is now dominated by “D” => eliminate “A”

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<td>0, 5</td>
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- In this “reduced game,” for “Player 2” the strategy of “e” is now dominated by “d” => eliminate “e”

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<td>0, 5</td>
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• In this “reduced game,” for “Player 1” the strategies of “B” and “D” are dominated by “C” => eliminate “B” and “D”

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<td>Player 1</td>
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<tr>
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<td>2,12</td>
<td>1,6</td>
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• In this “reduced game,” for “Player 2” the strategies of “b” and “c” are dominated by “d” => eliminate “b” and “c”

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<th>b</th>
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<tr>
<td>Player 2</td>
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<tr>
<td>A</td>
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• In this “reduced game,” for “Player 1” the strategy of “E” is dominated by “C” => eliminate “E”

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• Via Iterated Elimination of Dominated Strategies, we have identified the strategy pair of “C” for “Player 1” and “d” for “Player 2” as the unique Nash Equilibrium of the game
Example for which “IEDS” cannot solve the game:
- But, not all games can be solved by this approach. Consider:

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
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<tbody>
<tr>
<td>Player 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>8 , 7</td>
<td>4 , 5</td>
</tr>
<tr>
<td>Bottom</td>
<td>2 , 1</td>
<td>6 , 3</td>
</tr>
</tbody>
</table>

- Neither player has a dominant strategy or a dominated strategy => cannot whittle away any strategies by IEDS
- But, this game is simple enough that we can see (by drawing our “best response arrows”) that there are actually multiple Nash Equilibria

In a ‘2x2 game,’ what do the ‘best response arrows’ reveal?
1. Whether a player has or does not have a dominant strategy
   - If all the arrows for a player “point in the same direction,” then the strategy associated with either the row or column to which they all point is a dominant strategy (since this indicates that the “best reply” for the player is the same, regardless of the strategy chosen by her rival)
   - If all the arrows for a player do NOT “point in the same direction,” then the player does not have a dominant strategy (since this indicates that the “best reply” for the player depends upon the strategy chosen by her rival)

2. Whether a pair of strategies is or is not an equilibrium
   - If for a particular “cell” all the arrows “point inward,” then the pair of strategies leading to this cell is a Nash Equilibrium (since this indicates that no player could increase her own payoff by choosing a different strategy)
   - If for a particular “cell” any arrows “point outward,” then the pair of strategies leading to this cell is NOT a Nash Equilibrium (since this indicates that at least one player could increase her payoff by choosing a different strategy)
Ways to potentially maintain cooperation in a Prisoners’ Dilemma:

1. Resolving the dilemma by “Repeated Interaction”
   - Consider the following Prisoners’ Dilemma:
     
     \[
     \begin{array}{c|cc}
     & \text{Cooperate} & \text{Defect} \\
     \hline
     \text{Cooperate} & 10, 10 & 2, 12 \\
     \text{Defect} & 12, 2 & 4, 4 \\
     \end{array}
     \]
   - Suppose this game is repeated an uncertain number of times. After each period, the players will meet again with probability \( p \).
   - Recognize that in a repeated game, strategies can be very complex, since at any point in time a player can condition his choice on anything that has previously been observed. Examples of strategies in this repeated Prisoners’ Dilemma:
     - always choose cooperate in every period
     - always choose defect in every period
     - choose cooperate in every “even period” and defect in every “odd period”
     - in each period play whatever strategy my opponent played last period (tit-for-tat)
     - always choose cooperate so long as my opponent has chosen cooperate in every previous period, but once my opponent chooses defect then choose defect from in every single future period (grim trigger)
   - If my opponent announces she will use “grim trigger,” what is my best reply?
   - If I choose cooperate in every period, then my expected payoff is:
     \[
     \pi_c = 10 + 10p + 10p^2 + 10p^3 + 10p^4 + ...
     \]
   - If I instead choose defect in every period, then my expected payoff is:
     \[
     \pi_d = 12 + 4p + 4p^2 + 4p^3 + 4p^4 + ...
     \]
   - Always choosing cooperate is better if and only if:
     \[
     \pi_c \geq \pi_d \\
     10 + 10p + 10p^2 + 10p^3 + 10p^4 + ... \geq 12 + 4p + 4p^2 + 4p^3 + 4p^4 + ... \\
     6p + 6p^2 + 6p^3 + 6p^4 + ... \geq 2 \\
     p + p^2 + p^3 + p^4 + ... \geq \frac{2}{6} = \frac{1}{3}
     \]
   - Define \( p + p^2 + p^3 + p^4 + ... = \sum_{i=1}^{\infty} p^i \equiv A \)
   - Recognize that \( Ap = p^2 + p^3 + p^4 + p^5 + ... \)
   - Thus, \( A - Ap = (p + p^2 + p^3 + ...) - (p^2 + p^3 + p^4 + ...) = p \)
   - It follows that \( A(1 - p) = p \iff A = \frac{p}{1-p} \)
   - For this game always choosing cooperate is a best reply to grim trigger (in which case repetition can sustain cooperation) so long as
     \[
     \frac{p}{1-p} \geq \frac{1}{3} \iff 3p \geq 1 - p \iff 4p \geq 1 \iff p \geq \frac{1}{4}
     \]
2. **Resolving the dilemma by “Legal Restrictions”**

- If the two players can legally agree (ahead of time) to both “cooperate,” then clearly the dilemma can be resolved. But, such open collusion between firms if often illegal in most countries.
- However, U.S. cigarette manufacturers were able to resolve a dilemma by legal restriction in the early 1970s.
- Consider the following Prisoners’ Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>Philip Morris</th>
<th>Reynolds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advertise on TV</td>
<td>10, 20</td>
<td>5, 50</td>
</tr>
<tr>
<td>No TV ads</td>
<td>40, 15</td>
<td>35, 45</td>
</tr>
</tbody>
</table>

- The simple fact that these firms were engaged in a “repeated prisoner’s dilemma” was not enough to “resolve the dilemma” prior to 1971. However, on January 2, 1971 they were able to “resolve the dilemma” and no longer advertise on TV.
- Firms were able to realize larger profits in 1971 after “resolving the dilemma” => great news for cigarette manufacturers!
- How were they able to do this?
  - Starting on 1/2/71, there was a legal agreement which made it that firms would no longer advertise on TV.
  - How could the Government “let this happen”!?  
  - The government was the one who “made it happen” => on 1/2/71 the “Public Health Cigarette Smoking Act” (which made it illegal to advertise cigarettes on TV and radio) went into effect.
Determining a Mixed Strategy Equilibrium:

- Consider the following game:

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Top</td>
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</tr>
<tr>
<td>Bottom</td>
<td>6,3</td>
<td>2,5</td>
</tr>
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</table>

- This game does not have any Pure Strategy Nash Equilibria
- Consider the “Mixed Extension” of the game, in which
  - \( q \) denotes the probability with which “Player 2” chooses “Left” (so that \( 1-q \) denotes the probability with which “Player 2” chooses “Right”)
  - \( p \) denotes the probability with which “Player 1” chooses “Top” (so that \( 1-p \) denotes the probability with which “Player 1” chooses “Bottom”)
- Derive and graphically illustrate the “Best Reply Correspondence” for each player…”

Consider the choice by “Player 1” when “Player 2” chooses “Left” with probability \( q \) and “Right” with probability \( 1-q \) …

- Player 1’s expected payoff from choosing “Top” is: \( \pi_1^T = 4q + 10(1-q) = 10 - 6q \)
- Player 1’s expected payoff from choosing “Bottom” is: \( \pi_1^B = 6q + 2(1-q) = 2 + 4q \)
- Thus, Player 1’s payoff is strictly greater from choosing “Top” as opposed to “Bottom” (in which case his “best reply” is \( p = 1 \)) if and only if
  \[ \pi_1^T > \pi_1^B \iff q < \frac{8}{10} = \frac{4}{5} = .8 \]
- Further, Player 1’s payoff is strictly greater from choosing “Bottom” as opposed to “Top” (in which case his “best reply” is \( p = 0 \)) if and only if \( q > .8 \)
- Finally, Player 1 would realize the exact same expected payoff from choosing “Top,” “Bottom,” or any randomization between the two (in which case any value of \( 0 \leq p \leq 1 \) is a “best reply”) if and only if \( q = .8 \)
- Visually, the Best Reply Correspondence of Player 1 can be illustrated as:

Similarly, considering the choice by “Player 2” when “Player 1” chooses “Top” with probability \( p \) and “Bottom” with probability \( 1-p \) …

- Player 2’s expected payoff from choosing “Left” is: \( \pi_2^L = 7p + 3(1-p) = 3 + 4p \)
- Player 2’s expected payoff from choosing “Right” is: \( \pi_2^R = p + 5(1-p) = 5 - 4p \)
Thus, Player 2’s payoff is strictly greater from choosing “Left” as opposed to “Right” (in which case his “best reply” is \( q = 1 \)) if and only if
\[
\pi_2^L > \pi_2^R \iff p > \frac{2}{8} = \frac{1}{4} = .25
\]

Further, Player 2’s payoff is strictly greater from choosing “Right” as opposed to “Left” (in which case his “best reply” is \( q = 0 \)) if and only if \( p < .25 \)

Finally, Player 2 would realize the exact same expected payoff from choosing “Left,” “Right,” or any randomization between the two (in which case any value of \( 0 \leq q \leq 1 \) is a “best reply”) if and only if \( p = .25 \)

Visually, the Best Reply Correspondence of Player 2 can be illustrated as:

Drawing the two correspondences in the same graph, we have:

- If Player 1 chooses \( p^* = .25 \) and Player 2 chooses \( q^* = .8 \), then each player is choosing a mixed strategy that is a best reply to the strategy being played by his rival \( \Rightarrow \) this pair of mixed strategies is a Nash Equilibrium!
**Sequential Move Games:**
- Consider the following sequential move game:

  ![Game Tree Diagram]

  - Equilibrium strategies that are NOT “subgame perfect” => consider the following pair of strategies:
    - Firm 2 chooses “Maintain Collusion” if Firm 1 chooses “Don’t Enter Market B” and chooses “Price War” if Firm 1 chooses “Enter Market B” (i.e., Firm 1 threatens a price war if and only if Firm 2 enters…)
    - Firm 1 chooses “Don’t Enter Market B”
    - These pair of strategies can be illustrated as:
• This pair of strategies above does fit our definition of a Nash Equilibrium (...check all possible “unilateral deviations”)...), but it doesn’t seem reasonable! Why not?
• The “threat” by Firm 1 to engage in a “price war” following entry is NOT CREDIBLE => does not seem reasonable to support an equilibrium on such a threat that would clearly not be carried out if Player 1 were to ever “call the bluff” of Player 2
• For sequential move games, a “refinement” of Nash Equilibrium has been developed, in order to rule out such unreasonable equilibria => Subgame Perfect Nash Equilibrium
• So long as each player has a different payoff at each terminal node of the game, there will be exactly one SPNE for the game, which can be identified via “backward induction”
• First, consider the choice by “Firm 2” following each possible choice by “Firm 1”…

\[
\begin{array}{ccc}
\text{Firm 1} & \text{Firm 2} \\
\text{Don't enter "Market B"} & \text{Enter "Market B"} \\
\text{Firm 2} & \text{Firm 2} \\
\text{Maintain Collusion} & \text{Price War} & \text{Maintain Collusion} & \text{Price War} \\
\begin{pmatrix} 70 \\ 170 \end{pmatrix} & \begin{pmatrix} 40 \\ 100 \end{pmatrix} & \begin{pmatrix} 90 \\ 150 \end{pmatrix} & \begin{pmatrix} 50 \\ 110 \end{pmatrix}
\end{array}
\]
• Next, considering the initial choice by “Firm 1”…

Unique SPNE consists of the following pair of strategies (note: we must specify the optimal choice at every decision node, even those that are not reached during the play of the game)

• Firm 2 chooses “Maintain Collusion” if Firm 1 chooses “Don’t Enter Market B” and chooses “Maintain Collusion” if Firm 1 chooses “Enter Market B”
• Firm 1 chooses “Enter Market B”

• This solution is illustrated most directly by drawing arrows along each relevant branch of the game tree (again, indicate exactly one choice after every decision node) and circling the payoff vector at the realized outcome
Multiple Choice Questions:

1. Consider a two player game between “Player 1” and “Player 2.” “Player 1” has two available strategies: “Strategy A” and “Strategy B.” “Player 2” has three available strategies: “Strategy c,” “Strategy d,” and “Strategy e.” If “Strategy A” of “Player 1” is a “Best Reply” to a choice of “Strategy c” by “Player 2,” then
   A. “Strategy A” cannot be a dominant strategy for “Player 1.”
   B. “Strategy B” cannot be a dominant strategy for “Player 1.”
   C. “Player 1” must use “Strategy A” at any Nash Equilibrium of the game.
   D. More than one (perhaps all) of the above answers is correct.

2. Collusion refers to
   A. a game in which all players have a dominant strategy.
   B. a game in which all players completely disregard the impact of their own actions on their own payoff.
   C. an effort by firms to coordinate their actions in an attempt to increase both “total industry profit” and “individual profit of every firm” (compared to the profit levels which would result without any such coordination).
   D. a market structure in which firms are able to engage in First Degree Price Discrimination.

3. Which of the following statements corresponds to one of the “decision making rules of thumb” discussed in lecture?
   A. “If you have a dominant strategy, use it.”
   B. “If your rival has a dominant strategy, expect her to never use it.”
   C. “If you have a dominant strategy, recognize that your rival will expect that you will never use it.”
   D. More than one (perhaps all) of the above answers is correct.

4. Nash’s Existence Theorem states that
   A. individuals can only ever overcome a “Prisoners’ Dilemma” by repeated interaction.
   B. the unique equilibrium of any simultaneous move game can be determined by “Iterated Elimination of Dominated Strategies.”
   C. the unique Subgame Perfect Nash Equilibrium of any sequential move game can be determined by “backward induction.”
   D. every game with a finite number of players, each with a finite number of available pure strategies, has at least one Nash Equilibrium (potentially in “Mixed Strategies”).

5. Which of the following is NOT one of the “three basic elements of a game”? 
   A. Players.
   B. Rules of the Game.
   C. Strategies.
   D. More than one of the above answers is correct (since more than one of the above is NOT one of the “three basic elements of a game”).
6. A game that is played between the same players more than one time is called a
A. One-Shot Game.
B. Repeated Game.
C. Game of Incomplete Information.
D. Game of Complete Information.

7. The pioneering work “Theory of Games and Economic Behavior” was written by
A. Adam Smith.
B. John F. Nash, Jr. and Lloyd Shapley.
C. John Maynard Keynes.
D. John von-Neumann and Oskar Morgenstern.

8. In the “Mixed Extension” of a game,
A. the strategy chosen by each of your rivals should be viewed as entirely random.
B. the strategy choice of each player is allowed to be a “probability distribution” over his available pure strategies.
C. there can never be any Nash Equilibria.
D. players are never allowed to play “Pure Strategies.”

9. In a simultaneous move game with two players, it must always be the case that
A. the sum of the payoffs of both players is maximized at an outcome that is a Nash Equilibrium.
B. the sum of the payoffs of both players is minimized at an outcome that is a Nash Equilibrium.
C. there is at least one Nash Equilibrium (potentially in “mixed strategies”).
D. More than one (perhaps all) of the above answers is correct.

**Problem Solving or Short Answer Questions:**

1. Consider the 2 player simultaneous move game below:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Player 1</td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>80, 65</td>
</tr>
<tr>
<td>Bottom</td>
<td>70, 75</td>
</tr>
</tbody>
</table>

1A. Does this game fit the definition of a “Prisoner’s Dilemma”? Clearly explain why or why not.
1B. Determine all Nash Equilibria of this game.
2. Consider the following sequential move game:

Identify a pair of strategies that is a Nash Equilibrium but is not a Subgame Perfect Nash Equilibrium.

3. For each of the following games, determine if either player has a Dominant Strategy and identify all Pure Strategy Nash Equilibria.

3A. 

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>10 , 5</td>
</tr>
<tr>
<td>Bottom</td>
<td>8 , 7</td>
</tr>
</tbody>
</table>

3B. 

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>2 , 5</td>
</tr>
<tr>
<td>Bottom</td>
<td>8 , 1</td>
</tr>
</tbody>
</table>

3C. 

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>20 , 15</td>
</tr>
<tr>
<td>Bottom</td>
<td>6 , 0</td>
</tr>
</tbody>
</table>

3D. 

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>30 , 25</td>
</tr>
<tr>
<td>Bottom</td>
<td>20 , 45</td>
</tr>
</tbody>
</table>
4. Determine all Nash Equilibria of each of the following games.

4A.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>3, 1</td>
<td>1, 0</td>
</tr>
<tr>
<td>Bottom</td>
<td>2, 4</td>
<td>2, 5</td>
</tr>
</tbody>
</table>

4B.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>4, 9</td>
<td>2, 15</td>
</tr>
<tr>
<td>Bottom</td>
<td>0, 7</td>
<td>10, 3</td>
</tr>
</tbody>
</table>

4C.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>6, 10</td>
<td>15, 20</td>
<td>2, 9</td>
</tr>
<tr>
<td>Middle</td>
<td>12, 6</td>
<td>3, 8</td>
<td>5, 5</td>
</tr>
<tr>
<td>Bottom</td>
<td>4, 8</td>
<td>9, 0</td>
<td>10, 7</td>
</tr>
</tbody>
</table>

4D.

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>2, 4</td>
<td>5, 0</td>
<td>8, 3</td>
</tr>
<tr>
<td>Middle</td>
<td>0, 1</td>
<td>7, 3</td>
<td>4, 0</td>
</tr>
<tr>
<td>Bottom</td>
<td>1, 5</td>
<td>3, 2</td>
<td>6, 9</td>
</tr>
</tbody>
</table>

5. Consider the following simultaneous move game:

<table>
<thead>
<tr>
<th>Player 2</th>
<th>Left</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td>8, 8</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>Middle</td>
<td>0, 0</td>
<td>4, 4</td>
<td>0, 0</td>
</tr>
<tr>
<td>Bottom</td>
<td>0, 0</td>
<td>0, 0</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

For the Mixed Extension of this game, let \( p_1 \) denote the probability that Player 1 chooses “Top,” \( p_2 \) denote the probability that Player 1 chooses “Middle,” let \( p_3 = 1 - p_1 - p_2 \) denote the probability that Player 1 chooses “Bottom,” \( q_1 \) denote the probability that Player 2 chooses “Left,” \( q_2 \) denote the probability that Player 2 chooses “Center,” and \( q_3 = 1 - q_1 - q_2 \) denote the probability that Player 2 chooses “Right.”

5A. Identify all Pure Strategy Nash Equilibria.

5B. Is the strategy pair \((p_1, p_2, p_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) and \((q_1, q_2, q_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) a Nash Equilibrium? Explain why or why not.

5C. Is the strategy pair \((p_1, p_2, p_3) = (\frac{1}{2}, \frac{2}{3}, 0)\) and \((q_1, q_2, q_3) = (\frac{1}{2}, \frac{2}{3}, 0)\) a Nash Equilibrium? Explain why or why not.

5D. Based upon your answers to parts (5B) and (5C), identify a different pair of strategies that is a Mixed strategy Nash Equilibrium.
6. Consider the following three players sequential move game. In each payoff vector, the top number indicates the payoff of Player 1, the middle number indicates the payoff of Player 2, and the bottom number indicates the payoff of player 3. Determine the unique Subgame Perfect Nash Equilibrium of this game.

7. Consider the following two player simultaneous move game:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>$a$, $b$</td>
<td>$2$, $2$</td>
</tr>
<tr>
<td>Bottom</td>
<td>$2$, $2$</td>
<td>$c$, $d$</td>
</tr>
</tbody>
</table>

Specify values of $a$, $b$, $c$, and $d$ for which:

7A. “Top” is a dominant strategy for Player 1 and “Left” is a dominant strategy for Player 2.
7B. neither player has a dominant strategy and there are no Pure Strategy Equilibria.
7C. neither player has a dominant strategy and there are two Pure Strategy Equilibria.
7D. there are no Pure Strategy Equilibria and the unique Mixed Strategy Equilibrium is characterized by Player 1 choosing “Top” with probability $\frac{1}{5}$ and Player 2 choosing “Left” with probability $\frac{2}{5}$. 
8. Consider the following two player simultaneous move game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Top</td>
<td>12, 12</td>
</tr>
<tr>
<td>Bottom</td>
<td>20, 2</td>
</tr>
</tbody>
</table>

8A. Does this game fit the definition of a Prisoners’ Dilemma? Clearly explain.
8B. If this game is played only once, what outcome would you expect to observe?
8C. For the remainder of this question, suppose that this game will be repeated an uncertain number of times. After each period, the game will be played exactly one more time with probability \( p \). In this repeated game, if Player 2 were to follow a strategy of “always choose ‘Right’ in every single period,” what is the best reply of Player 1? Determine the expected value of Player 1’s payoff when he uses this best reply against Player 2’s strategy of “always choose ‘Right’ in every single period.”
8D. Continuing to suppose that this game will be repeated as described above, suppose that Player 2 is using the “grim trigger strategy” (i.e., choosing “Left” initially, and continuing to choose “Left” in each future period if and only if Player 1 has chosen “Top” in every single previous period). Determine Player 1’s expected payoff from choosing “Top” in every period against this strategy of Player 2. Determine Player 1’s expected payoff from choosing “Bottom” in every period against this strategy of Player 2. For what values of \( p \) can cooperation be maintained in this repeated game?

Answers to Multiple Choice Questions:
1. B
2. C
3. A
4. D
5. B
6. B
7. D
8. B
9. C

Answers to Problem Solving or Short Answer Questions:
1A. The definition of a “Prisoner’s Dilemma” was: a game in which every player has a dominant strategy (so that the game has a unique equilibrium characterized by all players using their dominant strategies), but in which there is some other outcome at which the payoff of every single player is higher than the equilibrium payoff.

For the game under consideration, both players do have dominant strategies (“Top” is the dominant strategy for Player 1 and “Left” is the dominant strategy for Player 2). However, when Player 1 plays “Top” and Player 2 plays “Left,” the realized
payoffs are (80) for Player 1 and (65) for Player 2. That is, each player is realizing the highest payoff that she could possibly realize. From here, it is clear that this is not a Prisoner’s Dilemma (since there is not “some other outcome at which the payoff of every single player is higher than the equilibrium payoff”).

1B. Since each player has a dominant strategy, the unique Nash Equilibrium is for each player to follow her dominant strategy. That is, the unique Nash Equilibrium is for Player 1 to play “Top” and for Player 2 to play “Left.”

2. By backward induction, the unique Subgame Perfect Nash Equilibrium is:

Recognize that at this outcome Player 2 does not realize her highest possible payoff of (200). Any Nash Equilibrium that is NOT Subgame Perfect must involve a player making a “non-credible threat/promise.” Suppose that Player 2 attempts to make such a non-credible commitment in order to realize her highest possible payoff of (200). More precisely, Player 2 uses the strategy of “choosing ‘left’ following a choice of ‘Left’ by Player 1” and “choosing ‘left’ following a choice of ‘Right’ by Player 1.” If Player 2 were to in fact play according to this strategy, then the best reply of Player 1 would be to choose “Left.” This pair of strategies is illustrated as:

This pair of strategies is an equilibrium, since neither player can increase her own payoff by changing only her choice at any of the decision nodes. But again, it is not a Subgame
Perfect Equilibrium, since it rests upon Player 2 making a sub-optimal choice in the Subgame following a choice of “Right” by Player 1.

3A. Player 1 has a dominant strategy of “Top” (it is a best reply for each available strategy of Player 2). Player 2 has a dominant strategy of “Left” (it is a best reply for each available strategy of Player 1). Thus, the unique equilibrium of the game is a Pure Strategy Nash Equilibrium in which Player 1 plays “Top” and Player 2 plays “Left.”

3B. Player 1 does not have a dominant strategy (“Bottom” is the best reply to “Left,” while “Top” is the best reply to “Right”). Player 2 has a dominant strategy of “Right” (it is a best reply for each available strategy of Player 1). Thus, the unique equilibrium of the game is a Pure Strategy Nash Equilibrium in which Player 2 plays “Right” (his dominant strategy) and Player 1 plays “Top” (her best reply to the dominant strategy of Player 2).

3C. Neither player has a dominant strategy. For Player 1, “Top” is the best reply to “Left,” while “Bottom” is the best reply to “Right.” For Player 2, “Left” is the best reply to “Top,” while “Right” is the best reply to “Bottom.” Thus, the game has two Pure Strategy Nash Equilibria – one in which Player 1 plays “Top” and Player 2 plays “Left,” and another in which Player 1 plays “Bottom” and Player 2 plays “Right.”

3D. Neither player has a dominant strategy. For Player 1, “Top” is the best reply to “Left,” while “Bottom” is the best reply to “Right.” For Player 2, “Right” is the best reply to “Top,” while “Left” is the best reply to “Bottom.” Thus, the game has no Pure Strategy Nash Equilibria.

4A. This game has two Pure Strategy Nash Equilibria as follows:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top 3 , 1</td>
<td>Right 1 , 0</td>
</tr>
<tr>
<td>Bottom 2 , 4</td>
<td>Right 2 , 5</td>
</tr>
</tbody>
</table>

Additionally, letting $p$ denote the probability with which Player 1 plays “Top” and letting $q$ denote the probability with which Player 2 plays “Left,” a Mixed Strategy Nash Equilibrium can be determined as:

$$
\pi^T_1 = 3q + (1 - q) = 2q + 2(1 - q) = \pi^a_1 \\
1 + 2q = 2 \\
q^* = \frac{1}{2}
$$

and

$$
\pi^L_2 = p + 4(1 - p) = 5(1 - p) = \pi^R_2 \\
4 - 3p = 5 - 5p \\
p^* = \frac{1}{2}
$$

4B. This game does not have any Pure Strategy Nash Equilibria:

<table>
<thead>
<tr>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
</tr>
<tr>
<td>Top 4 , 9</td>
</tr>
<tr>
<td>Bottom 0 , 7</td>
</tr>
</tbody>
</table>
However, by Nash’s Existence Theorem we know that there must exist a Mixed Strategy Equilibrium. Letting $p$ denote the probability with which Player 1 plays “Top” and letting $q$ denote the probability with which Player 2 plays “Left,” a Mixed Strategy Nash Equilibrium can be determined as:

$$
\pi_1^T = 4q + 2(1-q) = 10(1-q) = \pi_1^B
$$

$$
2q + 2 = 10 - 10q
$$

$$
q^* = \frac{2}{3}
$$

and

$$
\pi_2^L = 9p + 7(1-p) = 15p + 3(1-p) = \pi_2^R
$$

$$
2p + 7 = 12p + 3
$$

$$
p^* = \frac{2}{5}
$$

4C. Start by recognizing that for Player 2, “Right” is dominated by “Left.” Once we eliminate “Right” from consideration, we are left with the reduced game:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Left</td>
</tr>
<tr>
<td></td>
<td>Center</td>
</tr>
<tr>
<td>Middle</td>
<td>12 , 6</td>
</tr>
<tr>
<td>Bottom</td>
<td>4 , 8</td>
</tr>
</tbody>
</table>

In this reduced game, “Bottom” is dominated by “Top,” giving us:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Left</td>
</tr>
<tr>
<td></td>
<td>Center</td>
</tr>
<tr>
<td>Middle</td>
<td>12 , 6</td>
</tr>
<tr>
<td>Bottom</td>
<td>4 , 8</td>
</tr>
</tbody>
</table>

From here, recognize that in this reduced game “Left” is dominated by “Center,” so that we have:

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Center</td>
</tr>
<tr>
<td>Top</td>
<td>15 , 20</td>
</tr>
<tr>
<td>Middle</td>
<td>3 , 8</td>
</tr>
<tr>
<td>Bottom</td>
<td>9 , 0</td>
</tr>
</tbody>
</table>

In this reduced game, “Middle” is dominated by “Top.” Thus, the unique Nash Equilibrium of this game is for Player 1 to choose “Top” and Player 2 to choose “Center,” as illustrated below:
4D. Start by recognizing that for Player 1, “Bottom” is dominated by “Top.” Further, if we consider the reduced game in which Player 1 does not consider “Bottom,” then for Player 2 “Right” is dominated by “Left.” Eliminating “Bottom” for Player 1 and “Right” for Player 2 leaves us with the following reduced game:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>2, 4</td>
<td>5, 0</td>
</tr>
<tr>
<td>Middle</td>
<td>0, 1</td>
<td>7, 3</td>
</tr>
<tr>
<td>Bottom</td>
<td>1, 5</td>
<td>3, 2</td>
</tr>
</tbody>
</table>

First recognize that in this reduced game there are two Pure Strategy Nash Equilibria as illustrated below:

Further, considering the mixed extension of this reduced game (in which $p$ denotes the probability with which Player 1 plays “Top” and $q$ denotes the probability with which Player 2 plays “Left”), there is a Mixed Strategy Nash Equilibrium which can be identified as follows:

$$\pi^T_1 = 2q + 5(1 - q) = 7(1 - q) = \pi^M_1$$

$$5 - 3q = 7 - 7q$$

$$q^* = \frac{1}{7}$$

and

$$\pi^L_2 = 4p + (1 - p) = 3(1 - p) = \pi^C_2$$

$$3p + 1 = 3 - 3p$$

$$p^* = \frac{1}{3}$$

5A. There are three Pure Strategy Nash Equilibria as follows: one in which Player 1 chooses “Top” and Player 2 chooses “Left”; a second in which Player 1 chooses “Middle” and Player 2 chooses “Center”; and a third in which Player 1 chooses “Bottom” and Player 2 chooses “Right.”

5B. The strategy pair $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not a Nash Equilibrium. To see this, recognize that if Player 2 were to use $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then Player 1’s payoff from playing “Top” is $8\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) = \frac{8}{3}$, which is strictly greater than his payoff of playing either “Middle” (of $0\left(\frac{1}{3}\right) + 4\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) = \frac{4}{3}$) or “Bottom” (of $0\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) = \frac{2}{3}$). Thus, Player 1’s best reply would be $(p_1, p_2, p_3) = (1, 0, 0)$, not $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This alone implies that the given strategy pair is not a Nash Equilibrium.

5C. The strategy pair $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{1}{3}, 0\right)$ and $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ is a Nash Equilibrium. To see this, we must verify that each player is choosing a strategy that is a best reply to the strategy being played by his rival. When Player 2 uses $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, then Player 1
would realize a payoff of $8\left(\frac{1}{3}\right) + 0\left(\frac{2}{3}\right) + 0(0) = \frac{8}{3}$ from playing “Top,” a payoff of $0\left(\frac{1}{3}\right) + 4\left(\frac{2}{3}\right) + 0(0) = \frac{8}{3}$ from playing “Middle,” and a payoff of $0\left(\frac{1}{3}\right) + 0\left(\frac{2}{3}\right) + 2(0) = 0$ from playing “Bottom.” Thus, any strategy that plays “Bottom” with probability zero is a best reply for Player 1. The strategy $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ satisfies this criterion. Similarly, when Player 1 uses $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, then Player 2 would realize a payoff of $8\left(\frac{1}{3}\right) + 0\left(\frac{2}{3}\right) + 0(0) = \frac{8}{3}$ from playing “Left,” a payoff of $0\left(\frac{1}{3}\right) + 4\left(\frac{2}{3}\right) + 0(0) = \frac{8}{3}$ from playing “Center,” and a payoff of $0\left(\frac{1}{3}\right) + 0\left(\frac{2}{3}\right) + 2(0) = 0$ from playing “Right.” Thus, any strategy that plays “Right” with probability zero is a best reply for Player 2. The strategy $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ satisfies this criterion.

5D. Beyond the Mixed Strategy Nash Equilibrium given in part (5C), three other Mixed Strategy Nash Equilibria exist. Two of these are similar to that given in (5C), in that for each player they place positive weight on two of the three available strategies. These two equilibria are: Player 1 plays $(p_1, p_2, p_3) = \left(0, \frac{1}{3}, \frac{2}{3}\right)$ and Player 2 plays $(q_1, q_2, q_3) = \left(0, \frac{1}{3}, \frac{2}{3}\right)$; and Player 1 plays $(p_1, p_2, p_3) = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$ and Player 2 plays $(q_1, q_2, q_3) = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$. The final mixed strategy equilibrium involves each player randomizing over all three of his available strategies. More precisely, this equilibrium is: Player 1 plays $(p_1, p_2, p_3) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$ and Player 2 plays $(q_1, q_2, q_3) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$.

6. The unique Subgame Perfect Nash Equilibrium can be identified via backward induction as:
7A. For “Top” to be a dominant strategy for Player 1, we need \( a > 2 \) and \( c < 2 \). For “Left” to be a dominant strategy for Player 2, we need \( b > 2 \) and \( d < 2 \).

7B. For neither player to have a dominant strategy and for no Pure Strategy Equilibria to exist, we need the “best response arrows” to result in a “cycle.” One way to have this is for \( a > 2, b < 2, c > 2, \) and \( d < 2 \).

7C. For neither player to have a dominant strategy and for two Pure Strategy Equilibria to exist, we need the “best response arrows” to be such that either: “Top/Left” and “Bottom/Right” each have both arrows pointing inward; or “Top/Right” and “Bottom/Left” each have both arrows pointing inward. To have the former, we would need \( a > 2, b > 2, c > 2, \) and \( d > 2 \).

7D. Again, to have no Pure Strategy Equilibria we need the best response arrows to result in a “cycle.” As noted in (7B), one way to have this is with \( a > 2, b < 2, c > 2, \) and \( d < 2 \). In order for Player 1 to optimally choose a mixed strategy when Player 2 is playing “Left” with probability \( \frac{3}{2} \), we need:

\[
\frac{3}{2} a + \frac{1}{2} (2) = \frac{3}{2} (2) + \frac{1}{2} c
\]

\[
\Leftrightarrow \frac{3}{2} a - \frac{1}{2} c = \frac{3}{2}
\]

\[
\Leftrightarrow 2a - c = 2
\]

Similarly, in order for Player 2 to optimally choose a mixed strategy when Player 1 is playing “Top” with probability \( \frac{1}{2} \), we need:

\[
\frac{1}{2} b + \frac{1}{2} (2) = \frac{1}{2} (2) + \frac{1}{2} d
\]

\[
\Leftrightarrow \frac{3}{2} d - \frac{1}{2} b = \frac{3}{2}
\]

\[
\Leftrightarrow 4d - b = 6
\]

A set of values satisfying these restrictions is: \( a = 3, c = 4, b = 0, \) and \( d = \frac{3}{2} \).

8A. The definition of a “Prisoner’s Dilemma” was: a game in which every player has a dominant strategy (so that the game has a unique equilibrium characterized by all players using their dominant strategies), but in which there is some other outcome at which the payoff of every single player is higher than the equilibrium payoff. For Player 1, “Bottom” is a dominant strategy, and for Player 2, “Right” is a dominant strategy. When this pair is used, each player gets a payoff of (8). If instead Player 1 chose “Top” and Player 2 chose “Left,” then each player would get a payoff of (12). So, yes, this game does fit the definition of a Prisoners’ Dilemma.

8B. If the game is played only once, then each player should use her dominant strategy. We should expect Player 1 to play “Bottom,” Player 2 to play “Right,” and the players to each realize a payoff of (8).

8C. If Player 2 were to “always choose ‘Right’ in every single period” (regardless of the choices made by Player 1), then Player 1 should choose “Bottom” in every period. If this pair of strategies is chosen, then Player will realize a payoff of (20) in each and every period. Given the probability of having the repeated game played “one more time,” this outcome gives Player 1 an expected payoff of:

\[
\pi_1 = 20 + 20 p + 20 p^2 + 20 p^3 + 20 p^4 + ...
\]

\[
\pi_1 = 20 + 20 \sum_{i=1}^{\infty} p^i
\]
Recall that $\sum_{i=1}^{\infty} p^i = \frac{p}{1-p}$. Thus, the payoff of Player 1 in this case is

$$\pi_1 = 20 + 20 \frac{p}{1-p} = \frac{20}{1-p}.$$

8D. Suppose Player 2 uses the “grim trigger strategy.” If Player 1 chooses “Top” in every period, then she realizes a payoff of (12) in each period, giving her an expected payoff of

$$\pi_1 = 12 + 12p + 12p^2 + 12p^3 + 12p^4 + \ldots$$

$$\pi_1 = 12 + 12 \sum_{i=1}^{\infty} p^i$$

$$\pi_1 = 12 + 12 \left( \frac{p}{1-p} \right)$$

If instead Player 1 chooses “Bottom” in every period, then she realizes a payoff of (20) in the first period and a payoff of (8) in any subsequent period. This choice gives her an expected payoff of:

$$\pi_1 = 20 + 8p + 8p^2 + 8p^3 + 8p^4 + \ldots$$

$$\pi_1 = 20 + 8 \sum_{i=1}^{\infty} p^i$$

$$\pi_1 = 20 + 8 \left( \frac{p}{1-p} \right)$$

Cooperation (i.e., Player 2 chooses “Left” and Player 1 chooses “Top” in every period) can be maintained so long as:

$$12 + 12 \left( \frac{p}{1-p} \right) > 20 + 8 \left( \frac{p}{1-p} \right)$$

$$\iff 4 \left( \frac{p}{1-p} \right) > 8$$

$$\iff 4p > 8 - 8p$$

$$\iff 12p > 8$$

$$\iff p > \frac{8}{12} = \frac{2}{3}$$