A Risk Averse Seller in a Continuous Time Auction with a Buyout Option

Timothy Mathews*

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Abstract

An auction with a buyout option occurring over continuous time with rules similar to eBay’s “buy it now” option is analyzed. It is shown that a risk averse seller facing risk neutral bidders will choose a buyout price low enough so that the buyout option is exercised with positive probability in equilibrium. Further, when the seller is risk averse and bidders are risk neutral, allowing the seller to offer a buyout option results in an ex ante Pareto improvement, compared to a similar auction without such an option.

Keywords: Auctions, Internet, Buyout Option.

JEL classification: D44, L86, D8.

*Department of Economics, California State University-Northridge, 18111 Nordhoff St., Northridge, CA 91330-8374, USA; e-mail: tmathews@csun.edu; telephone: (818) 677-6696. I would like to thank Yair Tauman and Thomas Jeitschko for assistance. This paper is based upon Chapter Four of my doctoral dissertation.
1 Introduction

One way in which internet auction sites have changed the rules from those found in traditional auction markets is by offering a “buyout” or “auction stop” option. When such an option is available, any potential bidder can choose to stop the auction and purchase the item being auctioned at the buyout price, a pre-specified price chosen by the seller. The existence of such options on the internet was first noted by Lucking-Reiley (2000) who points out that such options in essence allow “the bidder to buy an early end to the auction by submitting a sufficiently high bid.”1 Internet auction sites offering such options include LabX, Mackley and Company, eBay, Sotheby’s, Bid or Buy, uBid, Amazon, MSN, and Yahoo!.2 As mentioned on the LabX web site, exercising this option allows the bidder to: get the item now (allowing the buyer to reduce costs associated with waiting for the auction to close, which could be weeks in the future); secure a pre-specified price (traditional bidding may be able to drive the final selling price above the buyout price); eliminate any risk of losing the auction to a bidder with a higher willingness to pay; and save monitoring costs associated with ongoing screening of the progress of the auction.

Buyout option rules differ across internet auction sites. LabX is a business-to-business agent auction site, specializing in the sale of used laboratory equipment. The auctions on LabX are ascending price auctions. A typical auction will last two to three weeks. The seller can choose to offer a buyout option, referred to as an “auction stop” option by LabX. In order to exercise this option, a bidder must submit an auction stop request to LabX. LabX then attempts to contact the seller. If the seller is successfully contacted at least 48 hours prior to the auction close, the seller will terminate the auction, and the item is sold at the buyout price to the bidder exercising the option.

1Page 245.
Mackley and Company is a merchant/agent site specializing in the sale of antique jewelry. Either traditional bids or proxy bids can be placed in auctions on this site. A bidder on this site can exercise the buyout option and receive the item at the “private purchase price,” an option which is only available before traditional bidding begins.

Industry leader eBay is a consumer-to-consumer agent auction site. Almost anything imaginable is sold on eBay. The auctions on eBay are ascending price auctions with proxy bidding. At the time when the item is initially posted, the seller can choose to offer a buyout option, an option referred to as “buy it now” by eBay. The buyout price is chosen by the seller, and the buyout option is available only before any bids are placed on the item. If the buyout option is exercised the auction ends immediately, with the bidder exercising the option receiving the item for certain at the buyout price. Thus, it is not only possible for this “temporary” option to be no longer available before the conclusion of the auction, but any potential bidder has a great deal of control over whether or not the option is available to other participants. The “buy now” option on Sothebys.com has rules similar to those of eBay’s “buy it now” option.

In addition to eBay, options of this nature are offered on five other large consumer to consumer agent sites: Bid or Buy (“buy now” option), uBid (“uBuyItNow”), Amazon (“take-it price”), MSN (“threshold price”), and Yahoo! (“buy price”). On each of these five sites, if a seller chooses to offer such an option, the option is available for the duration of the auction.

An important distinction should be made between a “permanent” buyout option, which is available for the duration of the auction, and a “temporary” buyout option, which may cease to be available before the conclusion of the auction. When available on eBay or Sotheby’s, the buyout option is only available so long as no traditional proxy bids have been placed. Thus, it is

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3Many internet auction sites allow for the submission of proxy bids. A proxy bid will “compete” at the minimum level necessary to be the leading bid. For a further explanation of the proxy bidding rules on eBay, see pages.ebay.com/help/buyerguide/bidding-prxy.html.
not only possible for the option to cease to be available before the conclusion of the auction, but any potential bidder has a great deal of control over whether or not the option is available to other participants. Similarly, on sites such as Mackley and LabX, the buyout option (if not exercised) will cease to be available at some point before the conclusion of the auction, a time which cannot be influenced by the behavior of the potential bidders. In contrast, if the buyout option is permanent, a bidder cannot alter the availability of the buyout option, short of exercising the option and as a result ending the auction.

Budish and Takeyama (2001) analyze an auction with a permanent buyout option. Modelling such an option as a maximum bid in an English auction, a situation in which two bidders with independent private valuations that take on one of two possible values is analyzed. When bidders are risk averse, an optimally set buyout price can increase the expected revenue of the seller. As a result, a risk neutral seller will find such an option beneficial.

Mathews (2002c) focuses on risk attitudes of participants in an auction with a temporary buyout option. If all agents are risk neutral, the seller will offer a buyout price high enough so that the option is never exercised. However, risk aversion on either side of the transaction can lead to the seller optimally offering a buyout price low enough so that the option is exercised with positive probability. Further, if bidders are risk neutral and the seller is risk averse, allowing the seller to offer a buyout option results in an ex ante Pareto improvement, compared to a second price sealed bid auction. It should be noted that this model does not capture some important aspects of the timing of eBay’s “buy it now” feature. Specifically, it does not account for the fact that a bidder exercising this option knows with certainty that he will receive the item. Further, while the model analyzed does have a temporary buyout option, it does not allow for the power that a bidder has on eBay of being able to make the buyout option no longer available to all other bidders.

Dennehly (2000) noted that one of the reasons eBay introduced this option
is that buyers wanted to be able to obtain and sellers wanted to be able to sell items more quickly. Mathews (2002b) analyzes an auction with a buyout option, allowing for such considerations by auction participants. A model of an auction with a buyout option occurring over continuous time, with rules mirroring those found on eBay, is developed. In particular, a bidder exercising the option will not only receive the item for certain, but any bidder can make an available option no longer available to all bidders by simply submitting a bid. Such an auction is analyzed allowing for time impatient auction participants that prefer a given transaction to occur sooner as opposed to later. When all market participants make no distinction as to when the transaction is finalized, the seller will choose a buyout price high enough so that the option is never exercised. However, time impatience on either side of the transaction can motivate the seller to choose a buyout price low enough so that the option is exercised with positive probability. Such an option results in an ex ante Pareto improvement, in comparison to a similar auction without a buyout option, if the seller is time impatient but the bidders are not.

This paper analyzes the model developed by Mathews (2002b), allowing for a seller that is risk averse. Observations from eBay are presented in Section 2 to illustrate the overall size and growth of this leading internet auction site, as well as the use of the buyout option on this site. A model of an auction with a buyout option occurring over continuous time, with rules similar to those on eBay, is presented in Section 3. Equilibrium behavior for risk neutral bidders is specified in Section 4. The subsequent choice of buyout price by the seller is analyzed in Section 5. It is shown that a risk averse seller in such an auction will choose a buyout price low enough so that bidders exercise the option with positive probability. The welfare of auction participants in an auction with a buyout option is examined in Section 6. When a risk averse seller faces risk neutral bidders, allowing the seller to offer a buyout option results in an ex ante Pareto improvement, compared to a similar auction without a buyout option. Section 7 concludes.
2 Observations from eBay

Observations from eBay are presented in this section in order to illustrate that not only is this site quite large and rapidly growing, but also the “buy it now” option is being used by auction participants. eBay, specializing in consumer-to-consumer commerce, hosted over 283 million auctions during the first half of 2002, resulting in sales in excess of $6.5 billion. Table 1 in Appendix B illustrates the recent growth of eBay. This table lists gross sales in U.S. Dollars and the number of auctions hosted during each quarter, as well as the number of registered users at quarter end, for each quarter since the fourth quarter of 1998.4

eBay’s “buy it now” feature has become quite popular since it was introduced in November 2000. A buyout option was available on approximately 30% of the items listed on eBay during the first quarter of 2001, 35% of the items listed on eBay during the second quarter of 2001, and 45% of the items listed on eBay during December 2001.5 Buyout options are also being widely used on other auction sites where they are available. For example, as of August 2001 a buyout option was offered on roughly 37% of the items listed for sale on Bid or Buy’s South Africa site.6

In order to gain more insight in to the behavior of auction participants when such an option can be offered, all items listed in the two eBay categories “Electronics; Electronic Games; Sony Playstation 2; Racing” and “Electronics; Electronic Games; Sony Playstation 2; Sports” on January 29, 2001 and January 30, 2001 were observed. These observations are summarized in Table 2 in Appendix B. These specific categories were chosen for observation because of the facts that the products within these categories are rather homogeneous7 and bidders are not likely to be able to realize substantial gains

4This data was compiled from eBay’s quarterly Financial Results, which are available online at www.shareholder.com/ebay/releases-earnings.cfm.
5These figures were obtained from eBay’s First Quarter 2001, Second Quarter 2001, and Fourth Quarter 2001 Financial Results.
6This figure was obtained from Nitzan Til, CEO of Bid or Buy South Africa.
7Homogeneous in terms of product “quality.” That is, even if an item was “used” it
from reselling the items.\footnote{This is a result of the fact that at this time the quantity supplied of these items far exceeded the quantity demanded in traditional retail outlets. This was because the corresponding hardware was not readily available during this time, due to underproduction on the part of Sony. Since bidders are unlikely to be able to realize substantial gains from resale, it does not seem unrealistic to suppose that they have independent private valuations for these items.}

During this period a total of 210 items were offered for sale in these two categories, 38 in the “Racing” category and 172 in the “Sports” category. “BIN avail” refers to the number of auctions in which a “buy it now” option was initially available. Overall, the “buy it now” option was available in 124, or over 59%, of the auctions. Of the 124 auctions where the option was available, it was exercised 34 times, or over 27% of the time it was offered. This is indicated in the column labelled “BIN exer.”

Recall that under the rules on eBay (where the “buy it now” option is no longer available after the first bid is placed), it is possible for the final selling price to exceed the “buy it now” price. The column “BINp” reports the average “buy it now” price when the option was available; “FNLp” reports the average final selling price. While the average final selling price is less than the average “buy it now” price, it is not uncommon for traditional bidding to drive the final selling price above the “buy it now” price. This is reported in the column “FNLp>BINp.” For the entire sample this occurs 21 times, or over 23% of the time that a “buy it now” option is available but not exercised.

3 A Model of eBay’s “Buy it Now” Option

The model outlined below was developed in Mathews (2002b). A situation in which \(n+1\) bidders, each with an independent private valuation \(v_i\), attempt to acquire a single object is analyzed. The model consists of a two stage game.
During the first stage the seller chooses a buyout price denoted $\bar{B}$. The second stage of the game consists of an ascending price auction with proxy bidding occurring over continuous time $T = [0, 1]$. During this auction, the buyout option is available so long as no bids have been submitted. Each potential bidder $i$ has an independent private valuation, $v_i \in [0, 1]$, which is determined before stage one, as well as an “arrival time”, $t_i \in [0, 1]$, which is realized during stage two. Bidder $i$ may “actively participate” in the auction during stage two at any time after his arrival time. If the buyout option is exercised, the transaction occurs at the time when the option is exercised. If the option is not exercised, and the item is sold by way of auction, the item is sold at time $t = 1$ to the bidder submitting the highest proxy bid for an amount equal to the second highest proxy bid submitted.

At any time $t$ a bidder that has arrived can observe the exact time at which any previous proxy bid was placed. Further, such a bidder can observe the amount of every proxy bid that has been submitted, with the exception of the proxy bid which is currently the highest. The exact amount of the proxy bid that is currently highest is observed by only the bidder that submitted the bid. Let $h(t)$ denote this information at time $t$. This observable history of play is an element of the set of possible histories at time $t$ denoted $H(t)$. Let $A_i(h(t))$ denote the set of possible actions for player $i$ to take as a function of the history of play $h(t)$. Bidder $i$ can only take an action at times after his own arrival. At any time after his arrival, bidder $i$ can: exercise the option (if it is available), place or update a proxy bid, or take no action. A strategy of bidder $i$ is a function that specifies an action for bidder $i$ to take at time $t$ from the set $A_i(h(t))$ for every possible $h(t) \in H(t)$ for every $t \in [0, 1]$.

If a bidder obtains the item, his payoff is equal to the difference between his valuation for the object and the amount he pays. A bidder not obtaining the item and making no payments realizes a payoff of zero. The payoff of the

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9By “actively participate” we mean submit (or update) a proxy bid or exercise the buyout option (if it is available). A bidder may increase, but not decrease, his proxy bid at any time before the close of the auction.
seller is the total amount of revenue which he receives. If the buyout option is exercised the seller receives $\bar{B}$. If the buyout option is not exercised, the item is sold to the bidder submitting the highest proxy bid for the amount of the second highest proxy bid submitted. The model proposed here closely resembles the eBay setup in that: the option is available so long as no traditional proxy bids have been placed; if a bidder exercises the option when it is available he will receive the item for sure; and a bidder can make the option no longer available to all bidders by placing a traditional proxy bid. This model is analyzed assuming that each $v_i$ is independently distributed $U[0,1]$ and each $t_i$ is independently drawn from a common continuous cumulative distribution function $G(t)$ such that $G(0) = 0$ and $G(1) = 1$.

Mathews (2002b) analyzes this model with risk neutral auction participants that are time impatient. Here it is assumed that auction participants make no distinction as to when the transaction occurs. Further, it is assumed that bidders are risk neutral, but the seller is risk averse.

The game is solved recursively. First, equilibrium behavior is identified for bidders in the second stage of the game as a function of $\bar{B}$. Once this is done, the choice of $\bar{B}$ by a strategic seller is examined, assuming that bidders follow the strategies identified as an equilibrium in stage two.

The first goal is to determine if a symmetric equilibrium exists such that in stage two every bidder $i$ follows a strategy of the form: if the option is available at time $t_i$, exercise the option immediately if $\bar{B} \leq B(v_i)$ and submit a proxy bid of $b_i = v_i$ immediately if $\bar{B} > B(v_i)$, for some prespecified function $B(v)$; if the option is not available at time $t_i$, submit a proxy bid of $b_i = v_i$ immediately. Let $\hat{\sigma}$ denote the set of strategies of this form.

An equilibrium of this form specifies strategies for potential bidders to follow which are essentially independent of the history of play $h(t)$. This results from the fact that by the time bidder $i$ observes any action by another bidder, there is either no available action for bidder $i$ to take or bidder $i$ has a weakly dominant strategy. That is, if bidder $i$ observes another bidder exercising the option, then the auction has ended and it is too late for bidder
i to take any action, and if bidder i observes another bidder submitting a proxy bid, then bidder i has a weakly dominant strategy of submitting a proxy bid of \( b_i = v_i \) immediately.

If \( B(v) \) is strictly increasing on the range \( v \in [0, 1) \), then let \( \bar{v} \) be the value of \( v \in [0, 1] \) such that \( B(\bar{v}) = B \). If \( B(v) < B \) for all \( v \in [0, 1] \), then let \( \bar{v} = 1 + \varepsilon \) with \( \varepsilon > 0 \). In equilibrium, a bidder should exercise the option immediately upon arrival if the option is available and \( v_i \geq \bar{v} \).

4 Risk Neutral Bidders

The following proposition specifies equilibrium strategies for bidders during stage two. All results are proved in Appendix A.

Proposition 1 (Mathews (2002b)) A symmetric equilibrium exists in which each bidder i: exercises the buyout option at time \( t_i \) if the option is available and \( \bar{B} \leq v_i - \frac{v_{n+1}}{n+1} \); and otherwise submits a proxy bid of \( b_i = v_i \) at time \( t_i \).

Let \( \sigma^* \) denote the \((n+1)\)-tuple of strategies described in the proposition above. When bidders follow \( \sigma^* \), it is simply the first bidder to arrive that determines whether or not the option is exercised. The option will be exercised if and only if the first bidder to arrive has \( v_i \geq \bar{v} \), where \( \bar{v} \) is such that \( B(\bar{v}) = \bar{B} \).

According to this strategy, a bidder will exercise an available option if the option price is “low enough,” with the function \( B(v) \) precisely characterizing what “low enough” means as a function of the bidder’s valuation. Examining the function \( B(v) = v - \frac{v_{n+1}}{n+1} \), it is clear that \( B(0) = 0 \) and \( B(1) = \frac{n}{n+1} \). These conditions imply that: if the buyout price is zero, then the first bidder to arrive will exercise the buyout option whatever his valuation happens to be, and if the buyout price is greater than \( \frac{n}{n+1} \), then no bidder will ever exercise the buyout option. Further, the option is exercised with positive probability
in equilibrium if and only if $\bar{B} < \frac{n}{n+1}$. Specifically, the option will be exercised with probability $Pr(E) = 1 - \bar{v}$. It is clear that $Pr(E)$ is decreasing in $\bar{v}$.

A finer understanding of bidder behavior can be obtained by examining the difference between the valuation of a bidder and the maximum buyout price for which the bidder is willing to exercise the buyout option. This difference is $\Lambda(v) = v - B(v)$. For $B(v) = v - \frac{v^{n+1}}{n+1}$ we have $\Lambda(v) = \frac{v^{n+1}}{n+1}$, which is simply the expected payoff for a bidder with valuation $v$ from competing in a similar auction without a buyout option. Thus, the first bidder to arrive will exercise the buyout option if his expected payoff from doing so exceeds his expected payoff from competing against $n$ bidders in an auction with proxy bidding. If his expected payoff from the auction is higher, he will submit a proxy bid (thereby making the option no longer available to all other bidders) and compete in the auction with proxy bidding. Finally, $\Lambda'(v) = v^n > 0$, indicating that the difference between a bidder’s valuation and the maximum price for which he is willing to exercise the buyout option is increasing in the valuation of the bidder.

Note that whenever the option is exercised with positive probability, the allocation of the item will be ex post inefficient with positive probability. In particular, the outcome will be ex post inefficient if the first bidder to arrive exercises the option, but does not have the highest valuation of all bidders. Under the assumption that each bidder has a valuation which is independently and identically distributed $U[0, 1]$, the probability of ex post inefficiency is

$$Pr(I) = \frac{n}{n+1} - \bar{v} + \frac{1}{n+1} \bar{v}^{n+1}.$$  

This expression is clearly decreasing in $\bar{v}$ for $\bar{v} \in [0, 1)$.

5 Choice of $\bar{B}$ by the Seller

Continuing to solve the game recursively, the seller’s choice of $\bar{B}$ is examined, assuming that bidders play according to $\sigma^*$ during stage two. The choice of $\bar{B}$ will first be analyzed for a risk neutral seller. Then the choice will be
analyzed for an expected utility maximizing risk averse seller.

5.1 Risk Neutral Seller

The following proposition addresses the choice of $\bar{B}$ by a risk neutral seller.

**Proposition 2 (Mathews (2002b))** When potential bidders use the strategies $\sigma^*$, any $\bar{B} \geq \frac{n}{n+1}$ maximizes the expected revenue of the seller. When $\bar{B} \geq \frac{n}{n+1}$ no bidder will exercise the buyout option.

A risk neutral seller is only concerned with expected revenue. When bidders follow the strategies $\sigma^*$ the expected revenue of the seller as a function of $\bar{v}$ is

$$\mu(\bar{v}) = \frac{2n\bar{v}}{n+1} - \bar{v}^2 + \frac{2\bar{v}^{n+2}}{(n+1)(n+2)}.$$  

Since $\mu'(\bar{v}) = \frac{2n}{n+1} - 2\bar{v} + \frac{2\bar{v}^{n+1}}{(n+1)}$ is positive for all $\bar{v} \in [0,1)$, the optimal choice by a risk neutral seller is $\bar{v}^* = 1$.

5.2 Risk Averse Seller

In many instances in internet auctions, the transactions which are occurring are consumer-to-consumer transactions. As a result, the seller is likely to be a risk averse individual, concerned with more than simply his expected revenue. The choice of buyout by such a risk averse seller is analyzed in this subsection.

Consider an agent with utility for realizations of revenue $x$ given by a Bernoulli utility function $u(x)$. For a distribution of realizations of revenue $F(x)$, the utility of the agent is given by the von-Neumann–Morgenstern expected utility function $U(F) = \int u(x) dF(x)$. When analyzing the choice of $\bar{B}$ by the seller, the decision becomes one of choosing the buyout price which results in the most desirable distribution of revenues.

Since $B'(v) > 0$ for $v \in [0,1)$, the choice of the seller can be examined as a choice of $\bar{v}$. Let $\bar{F}(x)$ denote the distribution of revenues which results
from the seller choosing a specific value of $v$. The class of such distributions will be referred to as feasible, since the seller can realize any distribution in this class by appropriately choosing the buyout price.

Rothschild and Stiglitz (1970) show that in general the preferences of an agent over different distributions of revenue cannot be stated as preferences over the mean and variance of the distributions alone. This can only be done if restrictive assumptions are made about either the Bernoulli utility function of the individual or the specific distributions of revenue from which the individual is choosing. Borch (1969) and Baron (1977) show that in general preferences can be stated over mean and variance alone only if the agent has a quadratic Bernoulli utility function. Others, such as Hanoch and Levy (1969) and Feldstein (1969), have attempted to show when it is valid to apply mean-variance analysis to restricted distributions of revenues. More recently, Bigelow (1993) sharply characterized classes of distributions of revenue over which preferences can be stated as preferences over mean and variance alone.

Bigelow notices that, “When mean-variance consistency for all risk-averse preferences has been obtained, it has been accomplished by restricting attention to a class of random variables having the property that within that class increases in variance were equivalent to increases in risk....If a single parameter is to index riskiness, then it must be possible to order all the lotteries in the class with respect to risk. Since there is an additional parameter to index expected value, this ordering need only be possible after suitable normalization.” Following Bigelow, consider the reflexive and transitive, but not complete, relation “is at least as risky in the sense of Rothschild and Stiglitz.” Bigelow defines a class of lotteries to be normalized risk comparable if the members of the lottery can be ordered by this relation after being normalized to have zero mean. If the set of lotteries under consideration can be so ordered after such a normalization, then all agents will agree on the relative riskiness of the different lotteries. Further, the variance completely

\[10\] Bigelow (1993), page 189.
indexes the relative riskiness of any lottery within this class. The main result of Bigelow is that for a class of lotteries that is closed under additive shifts, preferences over the mean and variance of the lotteries are consistent with preferences over the lotteries that can be represented by a Bernoulli utility function if and only if the set of lotteries is normalized risk comparable.

Let $x_{\bar{v}}$ be the random variable denoting the revenue of the seller resulting from $\bar{v} \geq 0$. Let $y_{\bar{v},c} = x_{\bar{v}} + c$ be the class of lotteries resulting from $\bar{v} \geq 0$ along with the addition of an arbitrary finite constant $c$. $y_{\bar{v},c}$ is closed under additive shifts. As the lemma below states, $y_{\bar{v},c}$ is normalized risk comparable. In order to simplify the proof of this result, the fact that the preferences of the seller are assumed to satisfy the independence axiom is used.

**Lemma 1** The class of lotteries $y_{\bar{v},c}$ is normalized risk comparable.

Since the distributions of revenue which result from choosing different $\bar{v}$ are normalized risk comparable, Bigelow’s result can be applied. Preferences over the mean and variance of the random revenues are consistent with preferences that can be represented by a Bernoulli utility function. This results from the fact that the distributions can be ranked in terms of riskiness, with the variance of the distribution as an index of the level of riskiness.

The seller must choose among the random variables $x_{\bar{v}}$. Let $\rho(\bar{v}) = E(x_{\bar{v}}^2) - E(x_{\bar{v}})^2$ denote the variance of the random variable $x_{\bar{v}}$.\(^\text{11}\) Differentiating $\rho(\bar{v})$ with respect to $\bar{v}$ leads to

$$\rho'(\bar{v}) = \frac{\partial E(x_{\bar{v}}^2)}{\partial \bar{v}} - 2\mu(\bar{v})\mu'(\bar{v}).$$

It has already been noted that

$$E(x_{\bar{v}}) = \mu(\bar{v}) = \frac{2n \bar{v}}{n + 1} - \bar{v}^2 + \frac{2\bar{v}^{n+2}}{(n + 1)(n + 2)},$$

\(^\text{11}\)The variance is denoted by $\rho$ instead of the standard notation of $\sigma^2$ to reinforce the fact that in this case variance completely indexes the riskiness of any possible distribution of revenues. Henceforth, $\rho(\bar{v})$ will refer to the riskiness of a distribution of revenues induced by $\bar{v}$, just as $\mu(\bar{v})$ refers to the expected revenue of such a distribution.
implying
\[ E(x_{\bar{v}})^2 = \left( \frac{2n\bar{v}}{n + 1} - \bar{v}^2 + \frac{2\bar{v}^{n+2}}{(n + 1)(n + 2)} \right)^2. \]

When the seller chooses \( B \leq \frac{n}{n+1} \), resulting in \( \bar{v} \in [0, 1] \), whether or not the buyout option is exercised depends only upon the valuation of the first bidder to arrive. Index the bidders based upon their realized values of \( t_i \) so that \( t_1 \leq t_2 \leq \ldots \leq t_n \leq t_{n+1} \). As a result, the first bidder to arrive arrives at time \( t_1 \) and has a valuation \( v_1 \sim U[0, 1] \). Thus, the expected value of the square of the revenue of the seller can be written as
\[
E(x_{\bar{v}}^2) = \int_0^{\bar{v}} \left[ \int_0^y x^2 g(x)dx + \int_y^1 \left\{ \int_0^y y^2 k(z)dz + \int_y^1 z^2 k(z)dz \right\} g(x)dx \right] h(y)dy + \int_{\bar{v}}^1 \bar{v}^2 h(y)dy
\]
where: \( y \) is the valuation of the first bidder to arrive; \( h(y) \) is the probability density function of \( y \); \( x \) is the largest valuation of the \( n \) bidders other than the first bidder to arrive; \( g(x) \) is the probability density function of \( x \); \( z \) is the second largest valuation of the \( n \) bidders other than the first bidder to arrive; and \( k(z) \) is the probability density function of \( z \) conditional upon a realized value of \( x \). Under the assumption that the valuation of each bidder is independently distributed \( U[0, 1] \), we have: \( h(y) = 1 \), \( g(x) = nx^{n-1} \), and \( k(z) = (n - 1) \left( \frac{1}{z} \right)^{n-1} z^{n-2} \). As a result,
\[
E(x_{\bar{v}}^2) = \frac{n(n - 1)\bar{v}}{(n + 1)(n + 2)} + \frac{2n\bar{v}^{n+2}}{(n + 1)(n + 2)} - \frac{2n\bar{v}^{n+3}}{(n + 2)(n + 3)} + (1 - \bar{v})B(\bar{v})^2.
\]
The derivative of this expression with respect to \( \bar{v} \) is
\[
D'(\bar{v}) = \frac{\partial E(x_{\bar{v}}^2)}{\partial \bar{v}} = G(\bar{v}) + H(\bar{v})
\]
with
\[
G(\bar{v}) = \frac{n(n - 1)}{(n + 1)(n + 2)} + \frac{2n\bar{v}^{n+1}}{n + 1} - \frac{2n\bar{v}^{n+2}}{n + 2}
\]
and
\[
H(\bar{v}) = 2(1 - \bar{v}) \left( \bar{v} - \frac{\bar{v}^{n+1}}{n + 1} \right) (1 - \bar{v}^n) - \left( \bar{v} - \frac{\bar{v}^{n+1}}{n + 1} \right)^2.
\]
Evaluating this expression at \( \bar{v} = 1 \), we have:

\[
D'(1) = \frac{n}{(n+1)^2(n+2)}
\]

which, along with \( \mu'(1) = 0 \), implies \( \rho'(1) = \frac{n}{(n+1)^2(n+2)} > 0 \). Further, since the lotteries induced by \( \bar{v} \) are normalized risk comparable, with those induced by lower \( \bar{v} \) being less risky, it follows that \( \rho'(\bar{v}) > 0 \) for \( \bar{v} \in (0,1) \).

By varying \( \bar{v} \in [0,1] \) the pair \((\rho(\bar{v}), \mu(\bar{v}))\) will change. The set of feasible \((\rho(\bar{v}), \mu(\bar{v}))\) combinations for the seller is illustrated in Figure 1 as a locus of points in \((\rho, \mu)\) space. A point in \((\rho, \mu)\) space is feasible if there exists \( \bar{v} \) such that the resulting \( x_{\bar{v}} \) has a mean of \( \mu(\bar{v}) \) and variance of \( \rho(\bar{v}) \).

If the seller were to choose \( \bar{v} \geq 1 \), the resulting point would be \((\rho(1), \mu(1)) = \left(\frac{2n}{(n+2)^2(n+3)}, \frac{n}{n+2}\right)\); if the seller were to choose \( \bar{v} = 0 \), the resulting point would be \((\rho(0), \mu(0)) = (0,0) \). Note that the slope of the locus of feasible points at a given point in \((\rho, \mu)\) space is \( \frac{\mu'(\bar{v})}{\rho'(\bar{v})} \). Since \( \mu'(\bar{v}) > 0 \) and \( \rho'(\bar{v}) > 0 \) for \( \bar{v} \in (0,1) \) this locus is upward sloping in \((\rho, \mu)\) space, with larger values of \( \bar{v} \) leading to larger values of both \( \rho \) and \( \mu \). \( \mu'(0) = \frac{2n}{n+1} \) and \( \rho'(0) = D'(0) = \frac{n(n-1)}{(n+1)(n+2)} \) imply that the slope of the locus of feasible \((\rho, \mu)\) pairs is equal to \( \frac{2(n+2)}{n-1} \) at the point induced by \( \bar{v} = 0 \). Similarly, \( \mu'(1) = 0 \) and \( \rho'(1) = \frac{n}{(n+1)^2(n+2)} \) imply that the locus of feasible \((\rho, \mu)\) pairs has a slope of zero at \((\rho(1), \mu(1))\). In order to see whether or not a seller will choose a buyout price which results in the option being exercised with positive probability in equilibrium, this locus will be closely examined around the point \((\rho(1), \mu(1))\).

It has already been shown that the distributions of payoffs arising from implementing a buyout option are normalized risk comparable. Further, a seller with preferences that can be represented by a continuous Bernoulli utility function \( u(x) \) will have an indifference curve passing through each feasible point in \((\rho, \mu)\) space. For a risk averse seller, the slope of each indifference curve must be strictly positive in \((\rho, \mu)\) space. The indifference curves passing through \((\rho(1), \mu(1))\) and \((\rho(0), \mu(0))\) for a risk averse seller are illustrated in Figure 2. Theorem 1, which characterizes the optimal buyout price for a risk averse seller, follows from these observations.
Theorem 1 When bidders use the equilibrium strategies $\sigma^*$, a risk averse seller maximizes his expected utility by choosing $\bar{B} < \frac{n}{n+1}$. When $\bar{B} < \frac{n}{n+1}$ the buyout option is exercised with positive probability.

As a result, a risk averse seller, regardless of the type or degree of risk aversion, will offer a buyout option that will be exercised with positive probability in equilibrium. That is, risk aversion on the part of the seller is a motivation for offering a buyout option in this model. Further, the optimal buyout price must be strictly positive. If the seller were to offer a buyout price of zero, he would guarantee himself a payoff of zero. Since any positive buyout price results in payoffs that are never strictly negative, but are strictly positive with positive probability, no seller will prefer a buyout price of zero to a strictly positive buyout price. Graphically, the indifference curve passing through $(\rho(0), \mu(0))$ must lie completely below the locus of feasible $(\rho, \mu)$ pairs. Letting $C_\bar{v}$ denote the certainty equivalent of the random revenue resulting from $\bar{v}$, this follows from the fact that $C_\bar{v} > 0$ for any $\bar{v} > 0$.

Something can be said about how the optimal choice of a risk averse seller would differ if the preferences of the seller were to differ. Start by considering a point $X = (\rho_X, \mu_X)$ in $(\rho, \mu)$ space. For a risk averse seller there exists an indifference curve in $(\rho, \mu)$ space representing other $(\rho, \mu)$ pairs that are just as desirable as $X$. For an arbitrary level of risk greater than $\rho_X$, say $\bar{\rho}$, there is some corresponding expected payoff $\bar{\mu}$, such that $(\rho_X, \mu_X) \sim (\bar{\rho}, \bar{\mu})$. Similarly, for an arbitrary level of risk less than $\rho_X$, say $\hat{\rho}$, there is some corresponding expected payoff $\hat{\mu}$, such that $(\rho_X, \mu_X) \sim (\hat{\rho}, \hat{\mu})$.

Now consider a seller that is more risk averse than this initial seller in that the second seller: requires a larger increase in expected payoff in order to be compensated for any fixed increase in the level of risk; and is willing to accept a larger decrease in expected payoff for any fixed reduction in the level of risk. That is, for a seller that is more risk averse, the value of $\bar{\mu}$ must be strictly larger for every $\bar{\rho} > \rho_X$, while the value of $\hat{\mu}$ must be strictly smaller for every $\hat{\rho} < \rho_X$. This simply states that when comparing the indifference
curves passing through any point $X = (\rho_X, \mu_X)$ for sellers with different risk attitudes, one seller is more risk averse than another if his indifference curve is such that in $(\rho, \mu)$ space it lies above the indifference curve of the second individual for levels of risk greater than $\rho_X$ and lies below the indifference curve of the second individual for levels of risk less than $\rho_X$.$^{12}$

Consider a risk averse seller $i$ optimally choosing a cutoff value $\bar{v}^*$ which results in the pair $X^* = (\rho^*, \mu^*)$. Let $A_i(X^*)$ denote the combinations $(\rho, \mu)$ such that $(\rho, \mu) \succ (\rho^*, \mu^*)$. Since $(\rho^*, \mu^*)$ is optimal for $i$, the intersection of this set with the locus of feasible $(\rho, \mu)$ pairs must be empty. Separate the set $A_i(X^*)$ into two sets $A^H_i(X^*)$ and $A^L_i(X^*)$, where $A^H_i(X^*)$ consists of points in $A_i(X^*)$ such that $\rho > \rho^*$ and $A^L_i(X^*)$ consists of points in $A_i(X^*)$ such that $\rho < \rho^*$. Clearly the intersection of $A^H_i(X^*)$ and the locus of feasible $(\rho, \mu)$ pairs is empty.

Consider a second seller $j$ that is more risk averse than the initial seller $i$. Let $A_j(X^*)$ denote the combinations of $(\rho, \mu)$ that $j$ prefers to $(\rho^*, \mu^*)$. Divide the set $A_j(X^*)$ into two sets $A^H_j(X^*)$ and $A^L_j(X^*)$, where $A^H_j(X^*)$ consists of points in $A_j(X^*)$ such that $\rho > \rho^*$ and $A^L_j(X^*)$ consists of points in $A_j(X^*)$ such that $\rho < \rho^*$. Since $j$ is more risk averse than $i$, $A^H_j(X^*) \subset A^H_i(X^*)$. As a result, the intersection of $A^H_j(X^*)$ and the locus of feasible $(\rho, \mu)$ pairs is empty. Further, $A^L_j(X^*) \subset A^L_i(X^*)$, implying that the intersection of $A^L_j(X^*)$ and the locus of feasible $(\rho, \mu)$ pairs is not necessarily empty.

Thus, seller $j$ (the more risk averse seller) will never optimally choose a pair $(\rho, \mu)$ with $\rho > \rho^*$, while he may choose a pair $(\rho, \mu)$ with $\rho < \rho^*$. With the number of bidders fixed, choosing a weakly lower level of risk corresponds to choosing a weakly lower cutoff value $\bar{v}^*$. This implies that, all other things equal, $\bar{v}^*$ is non-increasing in the level of risk aversion of the seller. Further, as the seller becomes more risk averse, the optimal buyout price weakly decreases, the probability with which the option is exercised weakly increases, and the probability of ex post inefficiency weakly increases.

$^{12}$This definition of more risk averse is developed, and the resulting implications on the optimal choice of agents are discussed, in Mathews (2002a).
6 Welfare of Auction Participants

A risk averse seller is clearly better off when he is allowed to offer a buyout option in such an auction. However, it is not immediately clear if bidders are made better off or worse off ex ante. Consider bidder $i$ before stage one. Such a bidder knows his own valuation, $v_i$, but does not know what value of $t_i$ he will realize and does not know the valuation or arrival time of any other bidder. As Theorem 2 below states, such a bidder is actually made weakly better off by allowing the seller to offer a buyout option (as opposed to having the seller sell the item by way of an ascending price auction with proxy bidding and no buyout option).

**Theorem 2** When bidders use the equilibrium strategies $\sigma^*$, allowing a risk averse seller to offer a buyout option results in an ex ante Pareto improvement, compared to an ascending price auction with proxy bidding and no buyout option.

Thus, not only does the seller benefit from being allowed to offer a buyout option, but allowing the seller to offer such an option also weakly increases the ex ante expected payoff of all bidders.

7 Conclusion

The focus of this paper is the behavior of auction participants in an auction with a temporary buyout option occurring over continuous time with rules mirroring those of eBay’s “buy it now” option. The model analyzed was developed in Mathews (2002b) to examine the behavior of time impatient auction participants. The primary focus here is the impact of risk aversion on the part of the seller in such an auction. It is shown that a risk averse seller facing risk neutral bidders will choose a buyout price low enough so that the buyout option is exercised with positive probability in equilibrium. Fur-
ther, allowing a risk averse seller to offer such an option results in an ex ante Pareto improvement, compared to a similar auction without a buyout option.

Appendix A

Proof of Proposition 1. The proof can be found in Mathews (2002b).

Q.E.D.

Proof of Proposition 2. The proof can be found in Mathews (2002b).

Q.E.D.

Proof of Lemma 1. Consider the class of lotteries $y_{\bar{v}, c} = x_{\bar{v}} + c$, where $c$ is an arbitrary finite constant. To see that the different distributions of revenue induced by $\bar{v}$, plus or minus a constant $c$, are normalized risk comparable, we need to first normalize the different random variables so that they each have mean zero. Let $\mu_y(\bar{v}, c)$ denote the expected value of $y_{\bar{v}, c}$. Clearly $\mu_y(\bar{v}, c) = \mu_x(\bar{v}) + c$, where $\mu_x(\bar{v})$ is the expected value of $x_{\bar{v}}$. Thus, $y_{\bar{v}, c} - \mu_y(\bar{v}, c) = x_{\bar{v}} - \mu_x(\bar{v})$. Define $z_{\bar{v}} = x_{\bar{v}} - \mu_x(\bar{v})$. Since $z_{\bar{v}}$ is independent of $c$, in order to show that the class of random variables $y_{\bar{v}, c}$ is normalized risk comparable, it is sufficient to show that the different random variables $x_{\bar{v}}$ are comparable in terms of risk. Consider two different random variables of the form $x_{\bar{v}}$, defined by different cutoff values $\hat{v}$ and $\bar{v}$ with $\hat{v} < \bar{v}$.

Think of the lottery $x_{\bar{v}}$ as being the combination of two different lotteries such that

$$x_{\bar{v}} = \bar{L}_{I} \Pr(V_1 \geq \hat{v}) + \bar{L}_{II} \Pr(V_1 < \hat{v})$$

with: $V_1$ being the valuation realized by the first bidder to arrive; $\bar{L}_{I}$ representing the random revenue that the seller receives if $V_1 \geq \hat{v}$; and $\bar{L}_{II}$ representing the random revenue that the seller receives if $V_1 < \hat{v}$.

Think of the lottery $x_{\bar{v}}$ as being a similar combination of two lotteries
such that

\[ x_\hat{v} = \hat{L}_I \Pr(V_1 \geq \hat{v}) + \hat{L}_{II} \Pr(V_1 < \hat{v}) \]

with: \( V_1 \) again being the valuation realized by the first bidder to arrive; \( \hat{L}_I \) representing the non-random payoff \( \hat{B} \) which is realized if \( V_1 \geq \hat{v} \) (and therefore exercises the option); and \( \hat{L}_{II} \) representing the random revenue that the seller receives if \( V_1 < \hat{v} \).

In the two compound lotteries above, \( \bar{L}_{II} = \hat{L}_{II} \). Thus, by the independence axiom, the only distinction between these two compound lotteries must be a distinction between the lotteries \( \bar{L}_I \) and \( \hat{L}_I \). This implies that \( x_\bar{v} \) and \( x_\hat{v} \) can be compared based upon riskiness if and only if \( \bar{L}_I \) and \( \hat{L}_I \) can be compared based upon riskiness. As noted above, the lottery \( \hat{L}_I \) represents the non-random payoff \( \hat{B} \) and therefore has an expected value of \( \hat{B} \). When this lottery is normalized to have mean zero, the resulting lottery is a non-random payoff of zero. The lottery \( \bar{L}_I \) represents a non-degenerate random payoff ranging from zero up to one. The expected value of the payoff from lottery \( \bar{L}_I \) is \( \mu(\bar{L}_I) \in (0, 1) \). After normalizing this lottery to have mean zero, the payoff from the normalized lottery ranges from \(-\mu(\bar{L}_I) < 0 \) to \( 1 - \mu(\bar{L}_I) > 0 \). As a result, the distribution functions of these normalized random variables are single crossing, implying that the normalization of lottery \( \bar{L}_I \) is second order stochastically dominated by the normalization of lottery \( \hat{L}_I \). This implies that the lottery \( \hat{L}_I \) is less risky than the lottery \( \bar{L}_I \), which in turn implies \( x_\hat{v} \) is less risky than \( x_\bar{v} \). Therefore, the class of lotteries defined by \( y_\bar{v} = x_\bar{v} + c \) is normalized risk comparable, with lower values of \( \bar{v} \) inducing distributions of revenue which are less risky. \( Q.E.D. \)

**Proof of Theorem 1.** Choosing \( \hat{B} < \frac{n}{n+1} \) corresponds to choosing \( \bar{v} < 1 \). So it must simply be shown that the seller will choose \( \bar{v} < 1 \).

If the seller chooses \( \bar{v} = 1 \), he realizes \((\rho(1), \mu(1))\). It has been shown that the slope of the locus of feasible bundles is equal to zero at \((\rho(1), \mu(1))\). Since the seller is risk averse, the slope of his indifference curve passing through \((\rho(1), \mu(1))\) must be strictly positive at \((\rho(1), \mu(1))\). As a result, for any
risk averse seller, there exists a feasible \((\rho(v), \mu(v))\) with \(v < 1\) such that \((\rho(v), \mu(v)) \succ (\rho(1), \mu(1))\). Thus, \(\bar{v} = 1\) cannot be optimal. \textit{Q.E.D.}

**Proof of Theorem 2.** It is straightforward to verify that for bidders with independent private valuations competing in an ascending price auction with proxy bidding and no buyout option, it is a dominant strategy for each bidder \(i\) to ultimately submit a bid of \(b_i = v_i\). Assume that all bidders follow this strategy in such an auction. The outcome of this auction is simply that the individual with the highest valuation receives the item and pays an amount equal to the second highest valuation of all bidders. This is the same outcome as a standard sealed bid second price auction when bidders bid truthfully. Thus, the expected payoff of a bidder \(i\) with valuation \(v_i\) in such an auction is

\[
\pi_{iA} = \int_0^{v_i} (v_i - y)ny^{n-1}dy = \frac{1}{n+1}v_i^{n+1}.
\]

Suppose the seller is now allowed to offer a buyout option and potential bidders play according to \(\sigma^*\). It has been shown that such a seller is strictly better off when choosing \(\bar{v} \in (0,1)\). Also, any bidder \(i\) with \(v_i \in [0, \bar{v})\) is just as well off with or without the buyout option available, since the outcome for such a bidder is the same in either case for every possible realization of valuations and arrival times of the \(n\) other bidders.

Now consider a bidder \(i\) with \(v_i \in [\bar{v}, 1]\). Such a bidder will exercise the option if it is available upon his arrival, and therefore has an ex ante expected payoff of

\[
\pi_{iB} = \frac{1}{n+1} (v_i - \bar{B}) + \frac{n}{n+1} \left( \frac{v_i^n \bar{v}}{n} - \frac{\bar{v}^{n+1}}{n(n+1)} \right)
\]

which simplifies to

\[
\pi_{iB} = \frac{1}{n+1} (v_i - \bar{v} + v_i^n \bar{v}).
\]

It must be shown that \(\pi_{iB} \geq \pi_{iA}\) for all \(v_i \in [\bar{v}, 1]\), and \(\pi_{iB} > \pi_{iA}\) for some \(v_i \in [\bar{v}, 1]\).
\[ \pi_iB \geq \pi_iA \iff (v_i^n - 1)(v_i - \bar{v}) \leq 0 \text{ and } \pi_iB > \pi_iA \iff (v_i^n - 1)(v_i - \bar{v}) < 0. \]

If \( v_i = \bar{v} \) or \( v_i = 1 \) then \( (v_i^n - 1)(v_i - \bar{v}) = 0 \). For \( v_i \in (\bar{v}, 1) \), \( (v_i^n - 1)(v_i - \bar{v}) < 0 \). Thus, ex ante no bidders are worse off and any bidder with \( v_i \in (\bar{v}, 1) \) is strictly better off when a risk averse seller is allowed to offer a buyout option. \textit{Q.E.D.}

## Appendix B

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\(^{13}\)In this table, (m) denotes “million” and (b) denotes “billion.”
References


FIGURE 1

Feasible \((\rho, \mu)\) for seller facing bidders using the strategies \(\sigma^*\)
FIGURE 2

Preferences of a risk averse seller