LINEAR OPTIMIZATION WITH PYTHON

José M. Garrido
Department of Computer Science

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College of Computing and Software Engineering
Kennesaw State University

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1 General Form of a Linear Optimization Model

The following is a general form of a linear optimization model that is basically organized in three parts.

1. The objective function, $f$, to be maximized or minimized, mathematically expressed as:

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$  \hspace{1cm} (1)

2. The set of $m$ constraints, which is of the form:

$$a_{i,1} x_1 + a_{i,2} x_2 + \ldots + a_{i,n} x_n \leq b_i \hspace{1cm} i = 1, \ldots m$$  \hspace{1cm} (2)

The other form is:

$$a_{i,1} x_1 + a_{i,2} x_2 + \ldots + a_{i,n} x_n \geq b_i \hspace{1cm} i = 1, \ldots m$$  \hspace{1cm} (3)

3. The sign restriction for variables: $x_j \geq 0$, or $x_j \leq 0$, or $x_j$ unrestricted in sign, $j = 1, \ldots n$.

Many problems are formulated with a mix of $m$ constraints with $\leq$, $=$, and $\geq$ forms. Note that the objective function, which is expressed mathematically in (1), and the constraints, which are expressed mathematically in (2) and (3), are linear mathematical (algebraic) expressions.

An important assumption included in the general formulation of a linear optimization problem is that the variables, $x_i, i = 1, \ldots n$, take numeric values that are real or fractional. In the case that one or more variables only take integer values, then other techniques and algorithms are used. These methods belong to the class of Integer Linear Optimization or Mixed Integer Optimization.

2 The Simplex Algorithm

The Simplex algorithm, due to George B. Dantzig, is used to solve linear optimization problems. It is a tabular solution algorithm and is a powerful computational procedure that provides fast solutions to relatively large-scale applications. There are many software implementations of this algorithm, or variations of it. The basic algorithm is applied to a linear programming problem that is in standard form, in which all constraints are equations and all variables non-negative.
2.1 Foundations of the Simplex Algorithm

For a given linear optimization problem, a point is the set of values corresponding to one for each decision variable. The feasible region for the problem, is the set of all points that satisfy the constraints and all sign restrictions. If there are points that are not in the feasible region, they are said to be in an infeasible region.

The optimal solution to a linear maximization problem is a point in the feasible region with the largest value of the objective function. In a similar manner, the optimal solution to a linear minimization problem is a point in the feasible region with the smallest value of the objective function.

There are four cases to consider in a linear optimization problem.

1. A unique optimal solution
2. An infinite number of optimal solutions
3. No feasible solutions
4. An unbounded solution

In a linear maximization problem, a constraint is binding at an optimal solution if it holds with equality when the values of the variables are substituted in the constraint.

2.2 Problem Formulation in Standard Form

Because the Simplex algorithm requires the problem to be formulated in standard form, the general form of the problem must be converted to standard form.

- For each constraint of \( \leq \) form, a slack variable is defined. For constraint \( i \), slack variable \( s_i \) is included. Initially constraint \( i \) has the general form:

\[
 a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \leq b_i \quad (4)
\]

To convert constraint \( i \) of the general form of the expression in (4) to an equality, slack variable \( s_i \) is added to the constraint, and \( s_i \geq 0 \). The constraint will now have the form:

\[
 a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n + s_i = b_i \quad (5)
\]
• For each constraint of ≥ form, an *excess variable* is defined. For constraint $i$, excess variable $e_i$ is included. Initially constraint $i$ has the general form:

$$a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \geq b_i \quad (6)$$

To convert constraint $i$ of the general form of the expression in (6) to an equality, excess variable $e_i$ is subtracted from the constraint, and $e_i \geq 0$. The constraint will now have the form:

$$a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n - e_i = b_i \quad (7)$$

Consider the following formulation of a numerical example:

Maximize: $5x_1 + 3x_2$

Subject to:

\[
\begin{align*}
2x_1 + x_2 &\leq 40 \\
2x_2 &\leq 50 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{align*}
\]

After rewriting the objective function and adding two slack variables $s_1$ and $s_2$ to the problem, the transformed problem formulation in standard form is:

Maximize: $z - 5x_1 - 3x_2 = 0$.

Subject to the following constraints:

\[
\begin{align*}
\frac{2x_1}{x_1} + \frac{x_2}{2x_2} + s_1 & = 40 \\
\frac{x_1}{x_1} + \frac{2x_2}{2x_2} + s_2 & = 50 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
s_1 &\leq 0 \\
s_2 &\leq 0
\end{align*}
\]

### 2.3 Generalized Standard Form

A generalized standard form of a linear optimization problem is:

Maximize (or minimize) $f = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the following constraints:
The constraints can be written in matrix form as follows:

\[
\begin{bmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & a_{m,2} & \cdots & a_{m,n}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix} =
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

Equation (9) can also be written as \(AX = B\), in which \(A\) is an \(m \times n\) matrix, \(X\) is a column vector of size \(n\), and \(B\) is a column vector of size \(m\).

### 2.4 Additional Definitions

To derive a basic solution to Equation (9), a set \(m\) of variables known as the basic variables is used to compute a solution. These variables are the ones left after setting the nonbasic variables, which is the set of \(n - m\) variables chosen and set to zero.

There can be several different basic solutions in a linear optimization problem. There could be one or more sets of \(m\) basic variables for which a basic solution cannot be derived.

A basic feasible solution to the standard formulation of a linear optimization problem is a basic solution in which the variables are non-negative.

The solution to a linear optimization problem is the best basic feasible solution to \(AX = B\) (or Equation (9)).

### 3 Description of The Simplex Algorithm

In addition to transforming the constraints to standard form, the expression of the objective function must be changed to an equation with zero on its right-hand side. The general expression:

\[
f = c_1x_1 + c_2x_2 + \cdots + c_nx_n
\]

is changed to
This equation becomes row 0 in the complete set of equations of the problem formulation. After this transformation the Simplex method can be used to solve the linear optimization problem.

### 3.1 General Description of the Simplex Algorithm

The following is a general description of the Simplex algorithm:

1. Find a basic feasible solution to the linear optimization problem; this solution becomes the initial basic feasible solution.

2. If the current basic feasible solution is the optimal solution, stop.

3. Search for an adjacent basic feasible solution that has a greater (or smaller) value of the objective function. An adjacent basic feasible solution has \( m - 1 \) variables in common with the current basic feasible solution. This becomes the current basic feasible solution, continue in step 2.

A linear optimization problem has an unbounded solution if the objective function can have arbitrarily large values for a maximization problem, or arbitrarily small values for a minimization problem. This occurs when a variable with a negative coefficient in the objective row (row 0) has a non-positive coefficient in every constraint.

### 3.2 Detailed Description of the Simplex Algorithm

A shorthand form of the set of equations known as the simplex tableau is used in the algorithm. Each tableau corresponds to a movement from one basic variable set BVS (extreme or corner point) to another, making sure that the objective function improves at each iteration until the optimal solution is reached. The following sequence of steps describes the application of the simplex solution algorithm:

1. Convert the LP to the following form:
   
   (a) Convert the minimization problem into a maximization one.
   (b) All variables must be non-negative.
   (c) All RHS values must be non-negative.
   (d) All constraints must be inequalities of the form \( \leq \).
2. Convert all constraints to equalities by adding a slack variable for each constraint.

3. Construct the initial simplex tableau with all slack variables in the basic variable set (BVS). The row 0 in the table contains the coefficient of the objective function.

4. Determine whether the current tableau is optimal. That is: If all RHS values are non-negative (called, the feasibility condition) and if all elements of the row 0 are non-positive (called, the optimality condition). If the answers to both questions are Yes, then stop. The current tableau contains an optimal solution. Otherwise, continue.

5. If the current basic variable set (BVS) is not optimal, determine, which non-basic variable should become a basic variable and, which basic variable should become a nonbasic variable. To find the new BVS with the better objective function value, perform the following tasks:

   (a) Identify the entering variable: The entering variable is the one with the largest positive coefficient value in row 0. (In case of a tie, the variable that corresponds to the leftmost of the columns is selected).

   (b) Identify the outgoing variable: The outgoing variable is the one with smallest non-negative column ratio (to find the column ratios, divide the RHS column by the entering variable column, wherever possible). In case of a tie the variable that corresponds to the upmost of the tied rows is selected.

   (c) Generate the new tableau: Perform the Gauss-Jordan pivoting operation to convert the entering column to an identity column vector (including the element in row 0).


At the start of the simplex procedure; the set of basis is constituted by the slack variables. The first BVS has only slack variables in it. The row 0 presents the increase in the value of the objective function that will result if one unit of the variable corresponding to the $j$th column was brought in the basis. This row is sometimes known as the indicator row because it indicates if the optimality condition is satisfied.

Criterion for entering a new variable into the BVS will cause the largest per-unit improvement of the objective function. Criterion for removing a variable from the current BVS maintains feasibility (making sure that the new RHS, after pivoting
remain non-negative). Warning: Whenever during the Simplex iterations you get a negative RHS, it means you have selected a wrong outgoing variable.

Note that there is a solution corresponding to each simplex tableau. The numerical of basic variables are the RHS values, while the other variables (non-basic variables) are always equal to zero. Note also that variables can exit and enter the basis repeatedly during the simplex algorithm.

3.3 Degeneracy and Convergence

A linear optimization problem (LP) is degenerate if the algorithm loops endlessly, cycling among a set of feasible basic solutions and never gets to the optimal solution. In this case, the algorithm will not converge to an optimal solution. Most software implementations of the Simplex algorithm will check for this type of non-terminating loop.

3.4 Two-Phase Method

The Simplex algorithm requires a starting basic feasible solution. The two-phase method can find a starting basic feasible solution whenever it exists. The two-phase simplex method proceeds in two phases, phase I and phase II. Phase I attempts to find an initial basic feasible solution. Once an initial basic feasible solution has been found, phase II is then applied to find an optimal solution to the original objective function.

The simplex method iterates through the set of basic solutions (feasible in phase II) of the LP problem. Each basic solution is characterized by the set of \( m \) basic variables \( x_{B1}, \ldots, x_{Bm} \). The other \( n \) variables are called nonbasic variables and denoted by \( x_{N1}, \ldots, x_{Nn} \).

4 Formulation of Linear Optimization Models

The formulating of a problem for linear constrained optimization is also known as linear optimization modeling or the mathematical modeling of a linear optimization problem (LP). Linear optimization modeling consists of four general steps, and these are as follows:

1. Identify a linear function, known as the objective function, to be maximized or minimized. This function is expressed as a linear function of the decision variables.

2. Identify the decision variables and assign to them symbolic names, \( x, y \), etc. These decision variables are those whose values are to be computed.
3. Identify the set of *constraints* and express them as linear equations and inequalities in terms of the decision variables. These constraints are derived from the given conditions.

4. Include the restrictions on the non-negative values of decision variables.

The objective function, \( f \), to be maximized or minimized is expressed by:

\[
f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_n x_n\tag{10}
\]

The set of \( m \) constraints are expressed in the form:

\[
a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \leq b_i \quad i = 1, \ldots m
\tag{11}
\]

Or, of the form:

\[
a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n \geq b_i \quad i = 1, \ldots m
\tag{12}
\]

The sign restrictions for variables are denoted by: \( x_j \geq 0 \), or \( x_j \leq 0 \), or \( x_j \) unrestricted in sign, \( j = 1, \ldots n \). Many problems are formulated with a mixed of \( m \) constraints with \( \leq, =, \) and \( \geq \) forms.

### 5 Example Problems

There are many real and practical problems to which the linear optimization modeling may be applied. The following examples, although very simple because they use only two variables, help to illustrate the general method involved in linear optimization modeling.

#### 5.1 Case Study 1

An industrial chemical plant produces two products, \( A \) and \( B \). The market price for a pound of \( A \) is $12.75, and that of \( B \) is $15.25. Each pound of substance \( A \) produced requires 0.25 lbs of material \( P \) and 0.125 lbs of material \( Q \). Each pound of substance \( B \) produced requires 0.15 lbs of material \( P \) and 0.35 lbs of material \( Q \). The amounts of materials available in a week are: 21.85 lbs of material \( P \) and 29.5 lbs of material \( Q \). Management estimates that at the most, 18.5 pounds of substance \( A \) can be sold in a week. The goal of this problem is to compute the amounts of substance \( A \) and \( B \) to manufacture in order to optimize sales.
5.1.1 Understanding the Problem

For easy understanding and for deriving the mathematical formulation of the problem, the data given is represented in a table as follows. As stated previously, the main resource required in the production of the chemical substances $A$ and $B$ are the amounts of material of type $P$ and $Q$.

<table>
<thead>
<tr>
<th>Material</th>
<th>Available</th>
<th>Substance of type A</th>
<th>Substance of type B</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>21.85</td>
<td>0.250</td>
<td>0.15</td>
</tr>
<tr>
<td>Q</td>
<td>29.50</td>
<td>0.125</td>
<td>0.35</td>
</tr>
</tbody>
</table>

5.1.2 Mathematical Formulation

Let $x_1$ denote the amount of substance (lbs) of type $A$ to be produced, and $x_2$ denote the amount of substance (lbs) of type $B$ to be produced. The total sales is $12.75x_1 + 15.25x_2$ (to be maximized). The objective function of the linear optimization model formulation of the given problem is:

Maximize $S = 12.75x_1 + 15.25x_2$

Subject to the constraints:

$$\begin{align*}
0.25x_1 + 0.15x_2 & \leq 21.85 \\
0.125x_1 + 0.35x_2 & \leq 29.5 \\
x_1 & \leq 18.5 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}$$

5.2 Case Study 2

A manufacturer of toys produces two types of toys: X and Y. In the production of these toys, the main resource required is machine time and three machines are used: M1, M2 and M3. The machine time required to produce a toy of type X is 4.5 hours of machine M1, 6.45 hours of machine M2, and 10.85 hours of machine M3. The machine time required to produce a toy of type Y is 7.25 hours of machine M1, 3.65 hours of machine M2, and 4.85 hours of machine M3. The maximum available machine time for the machines M1, M2, M3 are 415, 292 and 420 hours respectively. A toy of type X gives a profit of 4.75 dollars, and a toy of type Y gives a profit of 3.55 dollars. Find the number of toys of each type that should be produced to get maximum profit.

5.2.1 Understanding the Problem

For easy understanding and for deriving the mathematical formulation of the problem, the data given is represented in a table as follows. As stated previously, the
main resource required in the production of toys is machine time of machines M1, M2, and M3.

<table>
<thead>
<tr>
<th>Machine</th>
<th>Total time available</th>
<th>Req time toy type X</th>
<th>Req time toy type Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>415</td>
<td>4.5</td>
<td>7.25</td>
</tr>
<tr>
<td>M2</td>
<td>292</td>
<td>6.45</td>
<td>3.64</td>
</tr>
<tr>
<td>M3</td>
<td>420</td>
<td>10.85</td>
<td>4.85</td>
</tr>
</tbody>
</table>

5.2.2 Mathematical Formulation

Let $x$ denote the number of toys of type X to be produced, and $y$ denote the number of toys of the type Y to be produced. The total profit is: $= 4.75x + 3.55y$ (to be maximized). The objective function of the linear optimization model formulation of the given problem is:

Maximize: $P = 4.75x + 3.55y$

Subject to the constraints:

\[
\begin{align*}
4.5x + 7.25y & \leq 415 \\
6.45x + 3.65y & \leq 292 \\
10.85x + 4.85y & \leq 420 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

5.3 Case Study 3

A person needs to follow a diet that has at least 5,045 units of carbohydrates, 450.75 units of fat and 325.15 units of protein. Two types of food are available: $P$ and $Q$. A unit of food of type $P$ costs 2.55 dollars and a unit of food of type $Q$ costs 3.55 dollars. A unit of food of type $P$ contains 9.75 units of carbohydrates, 18.15 units of fat and 13.95 units of protein. A unit of food type $Q$ contains 22.95 units of carbohydrates, 12.15 units of fat and 18.85 units of protein. A mathematical linear model is needed to find the minimum cost for a diet that consists of a mixture of the two types of food and that meets the minimum diet requirements.

5.3.1 Understanding the Problem

For easy understanding and for deriving the mathematical formulation of the problem, the data given is represented in a table. For each type of food, the data include the cost per unit of food, and the contents, carbohydrates, fat, and proteins. The data for the diet problem is represented as follows:

<table>
<thead>
<tr>
<th>Food type</th>
<th>Cost</th>
<th>Carbohydrates</th>
<th>Fat</th>
<th>Protein</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>2.55</td>
<td>9.75</td>
<td>18.15</td>
<td>13.95</td>
</tr>
<tr>
<td>Q</td>
<td>3.35</td>
<td>22.95</td>
<td>12.15</td>
<td>18.85</td>
</tr>
</tbody>
</table>

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5.3.2 Mathematical Formulation

Let $x_1$ denote the amount of units of food type $P$ and $x_2$ the units of food type $Q$ contained in the diet. The total cost of the diet is: $2.55x_1 + 3.55x_2$. As stated previously, the main limitation is the lower bound requirement of carbohydrates, fat, and proteins. The combination of units of type $P$ and of type $Q$ should have the minimum specified units of carbohydrates, fat, and proteins. The objective function of linear optimization model formulation of the given diet problem is:

\[
\text{Minimize: } C = 2.55x_1 + 3.55x_2
\]

Subject to the following constraints:

\[
\begin{align*}
9.75x_1 + 22.95x_2 & \geq 5045 \\
18.15x_1 + 12.15x_2 & \geq 450.75 \\
13.95x_1 + 18.85x_2 & \geq 325.15 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

5.4 Case Study 4

The owners of a farm acquired a loan of $16,850.00 to produce three types of crops: corn, barley, and wheat, on 140 acres of land. An acre of land can produce an average of 135 bushels of corn, 45 of barley, or 100 bushels of wheat. The net profit per bushel of barley is $3.05, for corn is $1.70, and for wheat is $2.25. After the harvest, these crops must be stored in relatively large containers. At the present, the farm can store 3895 bushels. The total expenses to plant an acre of land is: $95.00 for corn, $205.00 for barley, and $115.00 for wheat. What amount of land should be the farm plan to dedicate to each crop in order to optimize profit?

5.4.1 Understanding the Problem

As in previous case studies, the data given is represented in a table for easy understanding and for deriving the mathematical formulation of the problem. There are three types of resources that will impose constraints on the problem formulation: the total storage capacity of the farm for the crops, the total available funds, and the total amount of land. The data for this problem is represented as follows:

<table>
<thead>
<tr>
<th>Resource</th>
<th>Total</th>
<th>Corn</th>
<th>Barley</th>
<th>Wheat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Storage (bushels)</td>
<td>3895</td>
<td>135</td>
<td>45</td>
<td>100</td>
</tr>
<tr>
<td>Funds ($)</td>
<td>16,850.00</td>
<td>95.00</td>
<td>205.00</td>
<td>115.00</td>
</tr>
<tr>
<td>Land (acres)</td>
<td>140</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
</tbody>
</table>
5.4.2 Mathematical Formulation

Let $x_1$ denote the amount of land in acres allotted to corn, $x_2$ the amount of land allotted to barley, and $x_3$ the amount of land dedicated to wheat. This problem is to optimize profit. The total net profit is denoted by $P$ and for each crop it consists of: net profit per bushel times the number of bushels per acre, times the amount of acres to plant. The constraints are derived from the resource limitations of the problem. The arithmetic expression for $P$ is given by:

$$P = 135 \times 1.70 \times x_1 + 45 \times 3.05 \times x_2 + 100 \times 2.25 \times x_3$$

The objective function of the linear optimization formulation of the given problem is:

Maximize: $P = 229.5x_1 + 137.25x_2 + 225.00x_3$

subject to the following constraints:

| 135$x_1$ | +45$x_2$ | +100$x_3$ | ≤ 3895  |
| 95$x_1$ | +205$x_2$ | +115$x_3$ | ≤ 16850.00 |
| $x_1$ | +$x_2$ | +$x_3$ | ≤ 140 |
| $x_1$ | 0 | 0 |
| $x_2$ | 0 | 0 |
| $x_3$ | 0 | 0 |

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